Technical Appendix to:

A Strategy for Opening a New Market and Encroaching on the Low End of the Existing Market

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Cheryl T. Druehl
Stonehill College
Easton, MA 02357
(508) 565-1463
cdruehl@gmail.com

Glen M. Schmidt
The David Eccles School of Business
University of Utah
Salt Lake City, UT
(801) 585-3160
glen.schmidt@business.utah.edu

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This technical appendix consists of the following:

APPENDIX A: Lemma 1, and Theorems 1 and 2.

APPENDIX B: Mapping of Disruptive Innovation to Low-end Encroachment, and Sustaining Innovation to High-end Encroachment.

APPENDIX C: Development of Opposite-Sloping Reservation Price Curves for Cell Phones.

APPENDIX A: Lemma 1, and Theorems 1 and 2

In this appendix we delineate the market outcomes when the new and old products are sold by two different firms (Theorem 1) or by the same firm (Theorem 2). Figure A-1 illustrates the results of Theorem 1 for z = 0.3 and k = 0.2 (Frame a), k = 0.5 (Frame b), and k = 0.8 (Frame c). Let $\pi_j = \left(p_j - c_j\right) q_j$ denote firm j's profit, $j \in \{O,\ \mathrm{N}\}$. In preparation for these theorems, we give Lemma 1, which establishes the quantities sold as a function of prices. Note that $\theta_j = \theta_j\left(p_O, p_N\right)$ but for simplicity we do not explicitly show the price dependencies. Proofs are given at the end of this Appendix.

Lemma 1. *Given p_O and p_N, sales quantities q_O and q_N are as follows:*

a) If $\theta_O \le \theta^* \le \theta_N$, then $q_O = \theta_O = 1 - p_O \ge 0$, and $q_N = z - \theta_N = \left(r_N - p_N\right)/k \ge 0$.

b) If $\theta_N \le \theta^* \le \theta_O$, then $q_O = \theta^* \ge 0$, and $q_N = z - \theta^* \ge 0$.

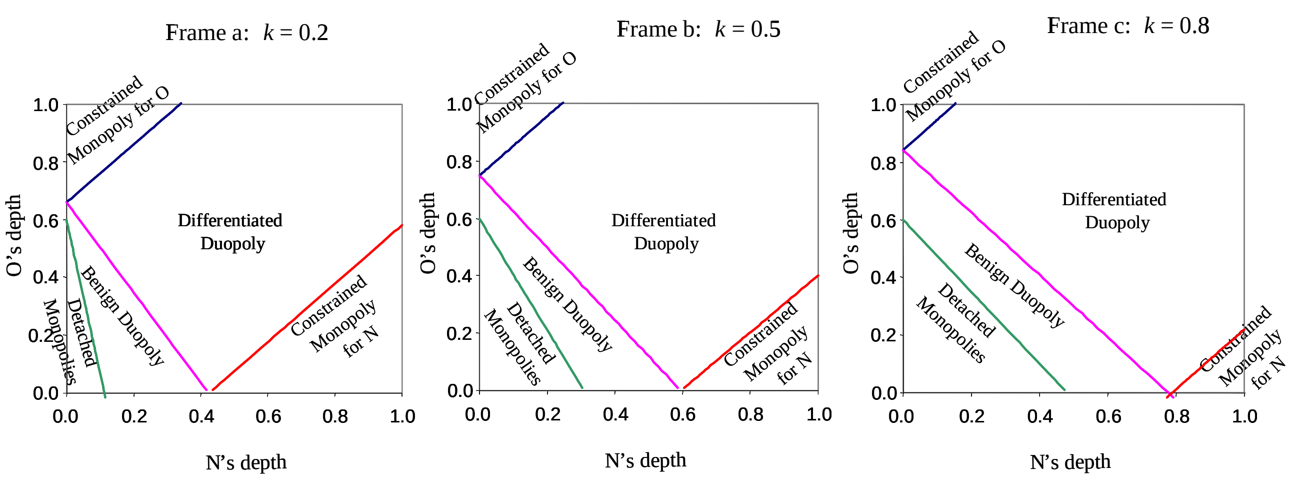

Figure A-1. Example of outcomes when products O and N are sold by different firms

Note in Figure A-1 that broad appeal is "good" – product N with broad appeal has an advantage in the sense that it can achieve a constrained monopoly "more easily" than product O. (Compare in Figure A-1 the intercept of the boundary to N's constrained monopoly region on the x-axis, which occurs at $z\left(1+2k\right)$, to the intercept of the boundary of O's constrained monopoly region on the y-axis, at $z\left(2+k\right)$, and note that $z\left(1+2k\right) < z\left(2+k\right)$.) The advantage of broad appeal is also illustrated by the way product N "reaches over" to encroach on product O earlier. (Compare the intercept of the boundary to the detached monopolies region on the x-axis, at $2kz$ with this boundary's intercept on the y-axis, at $2z$ and note that $2kz < 2z$.) If similar plots were made for other values of z, we would find that as z increases, the two constrained monopoly regions would shrink (and eventually disappear), while the detached monopolies region and the benign duopoly region would both grow toward the upper right, meaning the differentiated duopoly region would shrink.

Theorem 1. *When reservation price curves are opposite sloping and the two products are sold by different firms, the Nash equilibrium prices, quantities, and profits are as follows:*

	Detached Monopolies	***Benign Duopoly***
Conditions	$m_O \le 2z - m_N / k$.	$m_O \ge 2z - m_N / k$ *and* $m_O \le z(2+k) - m_N(2+k)/(1+2k)$.
Prices	$p_N = \frac{r_N + c_N}{2}$ *and* $p_O = \frac{1+c_O}{2}$.	*There is a continuum of equilibria for which prices are:* $p_N = r_N - kz + k(1-p_O)$ *and* $p_O = 1 - z + (r_N - p_N)/k$, *extending over the range:* $p_N \in \left[\frac{r_N + c_N}{2}, r_N - kz + k\frac{m_O}{2}\right]$ *and* $p_O \in \left[\frac{1+c_O}{2}, 1 - z + \frac{m_N}{2k}\right]$.
Quantities	$q_N = \frac{m_N}{2k}$ *and* $q_O = \frac{m_O}{2}$.	$q_N = z - \frac{1 - p_O + p_N - r_N + kz}{1+k}$ *and* $q_O = \frac{1 - p_O + p_N - r_N + kz}{1+k}$.
Profits	$\pi_N = m_N{}^2/(4k)$ *and* $\pi_O = m_O{}^2/4$.	$\pi_N = (p_N - c_N)q_N$ *and* $\pi_O = (p_O - c_O)q_O$.

	Differentiated Duopoly	***Constrained Monopoly for N***	***Constrained Monopoly for O***
Conditions	$m_O \ge m_N - z(1+2k)$ *and* $m_O \le m_N + z(2+k)$ *and* $m_O \le z(2+k) - m_N\frac{2+k}{1+2k}$.	$m_O \ge 2z - m_N / k$ *and* $m_O \le m_N - z(1+2k)$.	$m_O \ge 2z - m_N / k$ *and* $m_O \ge m_N + z(2+k)$.
Prices	$p_N = \frac{z(2+k) + r_N + 2c_N - m_O}{3}$ *and* $p_O = \frac{z(1+2k) + 1 + 2c_O - m_N}{3}$.	$p_N = r_N - kz - m_O$ *and* $p_O = c_O$.	$p_N = c_N$ *and* $p_O = 1 - z - m_N$.
Quantities	$q_N = \frac{z(2+k) - m_O + m_N}{3(1+k)}$ *and* $q_O = \frac{z(1+2k) + m_O - m_N}{3(1+k)}$.	$q_N = z$ *and* $q_O = 0$.	$q_N = 0$ *and* $q_O = z$.
Profits	$\pi_N = \frac{\left[m_N + z(2+k) - m_O\right]^2}{9(1+k)}$ *and* $\pi_O = \frac{\left[m_O - m_N + z(1+2k)\right]^2}{9(1+k)}$.	$\pi_N = (m_N - m_O - kz)z$ *and* $\pi_O = 0$.	$\pi_N = 0$ *and* $\pi_O = (m_O - m_N - z)z$.

Theorem 2. *When the reservation price curves are opposite sloping and the two products are sold by the same firm, the monopolist's profit maximizing prices, quantities, and profits are as follows:*

	Monopoly for O	***Monopoly for N***
Conditions	$m_O \geq 2z - m_N / k$ *and* $m_O \geq m_N + 2z$.	$m_O \geq 2z - m_N / k$ *and* $m_O \leq m_N - 2kz$.
Prices	$p_N \geq r_N$ *and* $p_O = 1 - z$.	$p_N = r_N - k\,z$ *and* $p_O \geq 1$.
Quantities	$q_N = 0$ *and* $q_O = z$.	$q_N = z$ *and* $q_O = 0$.
Profits	$\pi = \pi_N + \pi_O = \left(m_O - z\right) z$.	$\pi = \pi_N + \pi_O = \left(m_N - k\,z\right) z$.

	Joint Detached Monopoly	***Joint Covered Monopoly***
Conditions	$m_O \leq 2z - m_N / k$.	$m_O \geq 2z - m_N / k$ *and* $m_O \geq m_N - 2kz$ *and* $m_O \leq m_N + 2z$.
Prices	$p_N = \dfrac{r_N + c_N}{2}$ *and* $p_O = \dfrac{1 + c_O}{2}$.	$p_N = \dfrac{k\,m_O + \left(2+k\right) r_N + k\,c_N - 2k\,z}{2\left(1+k\right)}$ *and* $p_O = \dfrac{2k\left(1-z\right) + 1 + c_O + m_N}{2\left(1+k\right)}$.
Quantities	$q_N = \dfrac{m_N}{2k}$ *and* $q_O = \dfrac{m_O}{2}$.	$q_N = z - \dfrac{m_O - m_N + 2k\,z}{2\left(1+k\right)}$ *and* $q_O = \dfrac{m_O - m_N + 2k\,z}{2\left(1+k\right)}$.
Profits	$\pi = \pi_N + \pi_O = \dfrac{m_N^2}{4k} + \dfrac{m_O^2}{4}$.	$\pi = \pi_N + \pi_O = \dfrac{\left(m_N - m_O\right)^2 + 4\left(k\,m_O + m_N\right) - 4\,k\,z^2}{4\left(1+k\right)}$.

A picture showing the various market regions for Theorem 2 would be similar to Figure A-1 in that the joint detached monopoly region matches that of the detached monopolies. Further, the joint covered monopoly region is similar to the differentiated duopoly region in Theorem 1, and the one-product monopoly regions are similar to the one-product constrained monopoly regions in Theorem 1 except that the joint covered monopoly region abuts the joint detached monopoly region (there is nothing equivalent to the benign duopoly) and the boundaries between the joint covered monopoly region and one-product monopoly regions are parallel to the boundaries between the differentiated duopoly and the one-product constrained monopoly regions, but the *x* and *y*-axis intercepts are at 2 *k z* and 2 *z*, respectively.

Proof of Lemma 1: A customer of type $\theta \in \{0, \theta_O, \theta_N, \theta^*\}$ is allowed to buy both products, or both buy a product and buy nothing, since such a customer represents an infinitesimal purchase rate. Define $\psi_i(\theta)$ as the surplus for product *i* obtained by consumer type θ; in other words, the reservation price for product *i* held by consumer type θ minus the price for product *i*.

If $\theta_O \le \theta_N$, then $\theta_O \le \theta^* \le \theta_N$: this follows directly given $\theta_O = 1 - p_O$, $\theta_N = (p_N - r_N + kz)/k$, and $\theta^* = (1 - p_O + p_N - r_N + kz)/(1+k)$. Similarly, if $\theta_N \le \theta_O$, then $\theta_N \le \theta^* \le \theta_O$. Thus $\theta_O \le \theta^* \le \theta_N$ or $\theta_N \le \theta^* \le \theta_O$ (both sets of inequalities apply if $\theta_O = \theta_N$: in this case $\theta_O = \theta^* = \theta_N$). We show the proof for part b) of Lemma 1. The proof for a) follows similarly.

b) Given $\theta_N \le \theta_O$. If $\theta \in [0, \theta_N]$, then surplus $\psi_O(\theta) = 1 - \theta - p_O = \theta_O - \theta \ge \theta_O - \theta_N \ge 0$, and $\psi_N(\theta) = v_N + k\ \theta - p_N = k(\theta - \theta_N) \le 0$. Because $\psi_O(\theta) \ge \psi_N(\theta)$, the customer buys product *O*. If $\theta \in [\theta_N, \theta^*]$, then $\psi_O(\theta) = \theta_O - \theta \ge \theta_O - \theta^* \ge 0$ and $\psi_N(\theta) = k(\theta - \theta_N) \ge k(\theta_N - \theta_N) = 0$. Thus $\psi_O(\theta) - \psi_N(\theta) = \theta_O - \theta - k(\theta - \theta_N) \ge \theta_O + k\theta_N - (k+1)\theta^* \ge 0$, and the customer buys product *O*. If $\theta \in [\theta^*, \theta_O]$, $\psi_O(\theta) = \theta_O - \theta \ge \theta_O - \theta_O = 0$ and $\psi_N(\theta) = k(\theta - \theta_N) \ge k(\theta_N - \theta_N) = 0$. Thus $\psi_N(\theta) - \psi_O(\theta) \ge (k+1)\theta^* - k\theta_N - \theta_O = 0$, and the customer buys product *N*. If $\theta \in [\theta_O, z]$, then $\psi_O(\theta) \le 0$, and $\psi_N(\theta) \ge 0$, and the customer buys product *N*.

Proof of Theorem 1: We refer to the firms selling products *O* and *N* as firms *O* and *N*, respectively, and find below five possible equilibria.

Equilibrium 1: Detached Monopolies

If $\theta_O \le \theta_N$, then by Lemma 1a), $q_O = \theta_O = 1 - p_O$, and $q_N = z - \theta_N = z - [p_N - (r_N - kz)]/k$ and both firms sell strictly positive quantities. Given the other firm's price, Firm *O*'s and Firm *N*'s optimization problems are, respectively:

$$\max_{p_O} \quad \pi_O = (p_O - c_O) q_O = (p_O - c_O)(1 - p_O).$$

$$\max_{p_N} \quad \pi_N = (p_N - c_N) q_N = (p_N - c_N)\left(z - (p_N - r_N + kz)/k\right).$$

The objective functions are strictly concave such that the solutions are globally optimal for each firm. The solutions are:

$$p_O = \frac{1 + c_O}{2},\ q_O = \frac{m_O}{2},\ \text{and}\ \pi_O = \frac{m_O^{\ 2}}{4},\ \text{while}\ p_N = \frac{r_N + c_N}{2},\ q_N = \frac{m_N}{2k},\ \text{and}\ \pi_N = \frac{m_N^{\ 2}}{4k}.$$

Since these solutions are valid if $\theta_O \le \theta_N$, or $1 - p_O \le (p_N - r_N + kz)/k$, the above solutions hold if

$m_O \le 2z - m_N / k$. The market is not covered, except at the boundary $m_O = 2z - m_N / k$.

Equilibria 2-5 (The potential market is covered.)

If $\theta_O \ge \theta_N$, then by Lemma 1b), both firms sell weakly positive quantities, and the quantities sold by each firm are given in Lemma 1b). Given Firm *N*'s price, Firm *O*'s optimization problem is:

Max $\quad \pi_O = (p_O - c_O) q_O = (p_O - c_O)\theta^*$
p_O

subject to: $\quad \theta^* \ge 0$, $\Rightarrow 1 - p_O - (r_N - k\,z) + p_N \ge 0$: Nonnegative quantity of Product *O*.

$\theta^* \le z$, $\Rightarrow z(1+k) - (1 - p_O) - (p_N - (r_N - k\,z)) \ge 0$: Nonnegative quantity of Product *N*.

$\theta_O \ge \theta_N$, $\Rightarrow k(1 - p_O) + (r_N - k\,z) - p_N \ge 0$: Surplus must be ≥ 0 at θ^*.

The objective function is strictly concave: $\partial^2 \pi_O / \partial p_O{}^2 = -2/(1+k) < 0$. The second partial derivatives of the constraint functions are equal to zero: The constraint functions are also concave. Therefore any solution meeting the KKT conditions is globally optimal.

Let λ_O and λ_N denote the Lagrange multipliers associated with the non-negativity constraints for Product *O* and Product *N*, respectively, and let λ_S denote the Lagrange multiplier associated with the third (surplus) constraint. The Lagrangian becomes:

$$(p_O - c_O)\left(1 - p_O + p_N - (r_N - kz)\right)/(1+k) - \lambda_O\left(1 - p_O - (r_N - kz) + p_N\right)$$
$$-\lambda_N\left(z(1+k) - (1 - p_O) - \left(p_N - (r_N - kz)\right)\right) - \lambda_S\left(k(1 - p_O) + (r_N - kz) - p_N\right).$$

The KKT conditions are as follows, stemming from the first order conditions with respect to p_O, the orthogonality conditions, and the non-negativity conditions for the Lagrange multipliers.

$$\frac{1 - (r_N - kz) + p_N + c_O - 2p_O}{(1+k)} + \lambda_O - \lambda_N + k\lambda_S = 0, \tag{1}$$

$$-\lambda_O\left(1 - p_N - (r_O - kz) + p_O\right) = 0, \tag{2}$$

$$-\lambda_N\left(z(1+k) - (1 - p_O) - \left(p_N - (r_N - kz)\right)\right) = 0, \tag{3}$$

$$-\lambda_S\left(k(1 - p_O) + (r_N - kz) - p_N\right) = 0, \tag{4}$$

$$\lambda_O \ge 0, \tag{5}$$

$$\lambda_N \ge 0, \tag{6}$$

$$\text{and } \lambda_S \ge 0. \tag{7}$$

Similarly, the optimization problem for the firm selling product *N* is:

$$\max_{p_N} \quad \pi_N = \left(p_N - c_N\right) q_N = \left(p_N - c_N\right)\left(z - \theta^*\right)$$

$$= \left(p_N - c_N\right)\left(z - \frac{\left(1 - p_O\right) + \left(p_N - (r_N - kz)\right)}{\left(1 + k\right)}\right).$$

subject to: $\theta^* \geq 0$, $\Rightarrow 1 - p_O - (r_N - kz) + p_N \geq 0$: Nonnegative quantity of Product *O*.

$\theta^* \leq z$, $\Rightarrow z\left(1 + k\right) - \left(1 - p_O\right) - \left(p_N - (r_N - kz)\right) \geq 0$: Nonnegative quantity of Product *N*.

$\theta_O \geq \theta_N$, $\Rightarrow k\left(1 - p_O\right) + (r_N - kz) - p_N \geq 0$: Surplus must be ≥ 0 at θ^*.

The second derivative of the objective function is: $\partial^2 \pi_N \big/ \partial p_N{}^2 = -2\big/\left(1 + k\right) < 0$. Thus the objective function is concave. The second partial derivatives of the constraint functions equal zero: The constraint functions are also concave. Therefore any solution meeting the KKT conditions is globally optimal.

Let λ_O and λ_N denote the Lagrange multipliers associated with the non-negativity constraints for Product *O* and Product *N*, respectively, and let λ_S denote the Lagrange multiplier associated with the surplus constraint. The Lagrangian becomes:

$$\left(p_N - c_N\right)\left(z - \frac{\left(1 - p_O\right) + \left(p_N - (r_N - kz)\right)}{\left(1 + k\right)}\right) - \lambda_O\left(1 - p_O - (r_N - kz) + p_N\right)$$
$$-\lambda_N\left(z\left(1 + k\right) - (1 - p_O) - (p_N - (r_N - kz))\right) - \lambda_S\left(k\left(1 - p_O\right) + (r_N - kz) - p_N\right).$$

The KKT conditions are as follows, stemming from the first order conditions with respect to p_N, the orthogonality conditions, and the non-negativity conditions for the Lagrange multipliers.

$$z - \frac{1 - p_O - (r_N - kz) + 2p_N - c_N}{\left(1 + k\right)} - \lambda_O + \lambda_N + \lambda_S = 0, \tag{8}$$

$$-\lambda_O\left(1 - p_O - \left(r_N - kz\right) + p_N\right) = 0, \tag{9}$$

$$-\lambda_N\left(z\left(1 + k\right) - \left(1 - p_O\right) - \left(p_N - \left(r_N - kz\right)\right)\right) = 0, \tag{10}$$

$$-\lambda_S\left(k\left(1 - p_O\right) + \left(r_N - kz\right) - p_N\right) = 0, \tag{11}$$

$$\lambda_O \geq 0, \tag{12}$$

$$\lambda_N \geq 0, \tag{13}$$

$$\text{and } \lambda_S \geq 0. \tag{14}$$

As indicated, for each firm there are three constraints (and three associated Lagrange multipliers). This yields eight possible combinations of binding and non-binding constraints, but at most one constraint can be binding: if both non-negativity constraints were binding then both firms would realize zero sales, in which case $\theta_O \leq \theta_N$ which violates the condition $\theta_O \geq \theta_N$, while if a non-negativity constraint and the constraint for positive surplus were both binding this would imply that one firm realizes zero sales but also prices at the maximum reservation price (this could not be an equilibrium because the firm would always be willing to reduce its price down toward cost in an attempt to gain sales, and cost is assumed to be less than the maximum reservation price). This leaves four viable solutions, one involving no binding constraints and the other three involving the three respective individual constraints. The values of m_O and m_N establish the regions in which each solution applies, as identified below.

Equilibrium 2: Both Firms Sell Strictly Positive Quantities in a Differentiated Duopoly

When none of the three constraints are binding ($\lambda_O = \lambda_N = \lambda_S = 0$), the first order conditions give Firm *O*'s and Firm *N*'s reaction functions (for emphasis we list below each price as a function of the other price, but omit these dependencies elsewhere for brevity):

$$\text{Firm } O\text{:}\quad p_O\left(p_N\right) = \left(1 + c_O + p_N - \left(r_N - kz\right)\right)/2 \tag{15}$$

$$\text{Firm } N\text{:}\quad p_N\left(p_O\right) = \left(p_O + r_N + c_N + z - 1\right)/2 \tag{16}$$

Simultaneously solving (15) and (16) for prices yields: $p_O = \left[z\left(1+2k\right) + 1 + 2c_O - m_N\right]\big/3$ and $p_N = \left[z\left(2+k\right) + r_N + 2c_N - m_O\right]\big/3$. Substituting these into the equations for quantities and profits yields the expressions in Theorem 1 for the differentiated duopoly. The condition $\theta^* \geq 0$ is satisfied if and only if $m_O \geq m_N - z\left(1+2k\right)$, while $\theta^* \leq z$ is satisfied if and only if $m_O \leq m_N + z\left(2+k\right)$ and $\theta_O \geq \theta_N$ is satisfied if and only if $m_O \geq -\left(2+k\right)m_N / \left(1+2k\right) + z\left(2+k\right)$. (These are found by substituting the optimal prices into the constraint functions.) When these conditions are satisfied, the reaction functions $p_N\left(p_O\right)$ and $p_O\left(p_N\right)$ yield a dynamically stable tâtonnement process and a unique equilibrium. See Fudenberg and Tirole (1995), p. 24.

Equilibrium 3: Only Firm O Sells a Strictly Positive Quantity in a Constrained Monopoly

A solution when only the second constraint is binding, $\lambda_N \geq 0$ and $\lambda_O = \lambda_S = 0$, implies Firm *N* gets no sales and therefore no profits, $q_N = 0$ and $\pi_N = 0$, also indicating Firm *N* prices at cost, $p_N = c_N$ (similar to the Bertrand result of price competition, if Firm *N* priced above cost and sold nothing, Firm *N* would always be willing to reduce price to achieve positive sales). From the KKT conditions and constraints, we find this solution is an equilibrium if and only if $m_O \geq m_N + z(2+k)$ and $m_O \geq 2z - m_N/k$, yielding $p_O = 1 - z - m_N$, $q_O = z$, and $\pi_O = (m_O - m_N - z)z$.

Equilibrium 4: Only Firm N Sells a Strictly Positive Quantity in a Constrained Monopoly

A solution when only the first constraint is binding, $\lambda_O \geq 0$ and $\lambda_N = \lambda_S = 0$, implies Firm *O* gets no sales and therefore no profits, $q_O = 0$ and $\pi_O = 0$, also indicating Firm *O* prices at cost, $p_O = c_O$. From the KKT conditions and constraints, we find this solution is an equilibrium if and only if $m_O \leq m_N - z(1+2k)$ and $m_O \geq 2z - m_N/k$, yielding $p_N = r_N - kz - m_O$, $q_N = z$, and $\pi_N = (m_N - m_O - kz)z$.

Equilibrium 5: Both Firms Sell Strictly Positive Quantities in a Benign Duopoly

A solution when only the third constraint is binding, $\lambda_S \geq 0$ and $\lambda_O = \lambda_N = 0$, applies if $m_O \geq 2z - m_N/k$ and $m_O \leq z(2+k) - (2+k)m_N/(1+2k)$, per the KKT conditions and constraints. The surpluses equal zero at θ^*. The reaction functions are found from (4) and (11): $p_N(p_O) = k(1-p_O) + r_N - kz$ and $p_O(p_N) = 1 - z + [r_N - p_N]/k$, indicating the response functions are coincident and there are an infinite number of equilibria. Given these reaction functions and the KKT conditions and constraints, we find:

$r_N - k\,z \leq p_N \leq r_N$: As determined by substituting $p_O(p_N)$ into the conditions $0 \leq \theta^* \leq z$.

$1 - z \leq p_O \leq 1$: As determined by substituting $p_N(p_O)$ into the conditions $0 \leq \theta^* \leq z$.

These five cases have partitioned all possible values of m_O and m_N.

Proof of Theorem 2: The proof is similar to that of Theorem 1, but in this case the single firm maximizes its sum of profits from Products *O* and *N*.

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APPENDIX B: Mapping of Disruptive Innovation to Low-end Encroachment, and Sustaining Innovation to High-end Encroachment

This appendix shows disruptive innovation maps to low-end encroachment, and sustaining innovation maps to high-end encroachment. To do so we use the trajectory charts that Christensen (1997), Christensen and Raynor (2003), and Christensen et al. (2004) (abbreviated C, CR, and CAR, respectively) state form the basis for the theory of disruptive innovation. A trajectory chart for the disk drive industry is shown in C (p. 16), while similar charts are presented in CR (pp. 33 and 44), CAR (p. xvi) and Bower and Christensen (1995). A summary list of abbreviations used is shown in Table B-1 below.

Table B-1. Abbreviation of References (in the order first used)

C	Christensen (1997)
CR	Christensen and Raynor (2003)
CAR	Christensen et al. (2004)

The Mapping of Disruptive Innovation to Low-end Encroachment

The charts in CR/CAR are three-dimensional (3-D) in an attempt to show more information, but in turn they lose some of the richness of the original 2-D disk drive chart in C. To show both the richness of the original chart as well as the additional information conveyed in the 3-D charts, we develop the two frames shown in Figure B-1. (In their 3-D graphs they only show information at two points along the third dimension, so we can effectively show the same information with just two 2-D frames.) We add some further richness to the charts, which will be justified below when that information is needed.

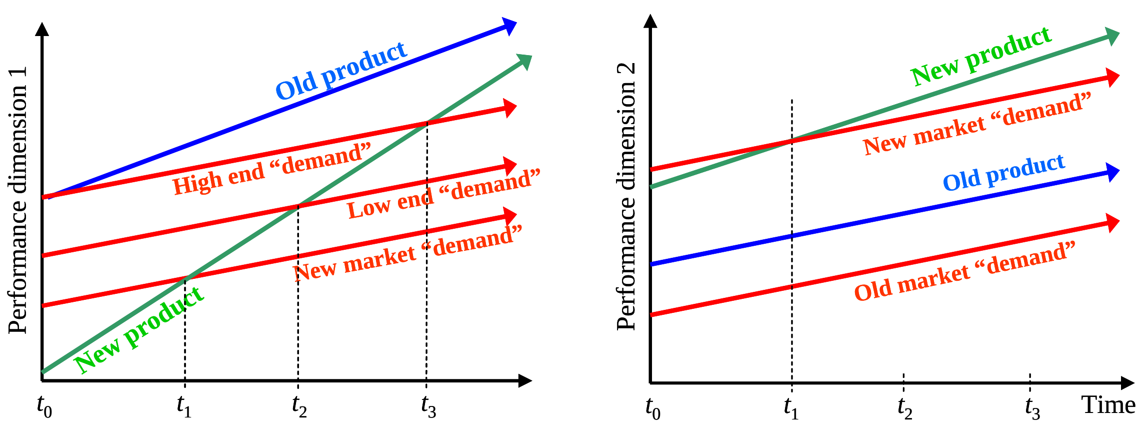

Figure B-1. Disruptive innovation theory (adapted from C, p. 16 and CAR p. xvi)

Disruptive innovation theory is described by C/CR/CAR assuming two performance dimensions. The left frame in Figure B-1 applies to the first dimension, the one most highly valued by current high-end customers, while the right frame applies to the second dimension, the one highly valued by a new market – that is, if there is a new market. CR distinguish between new-market disruptions which first open up a new market before encroaching on the old market, and low-end disruptions which are simply lower-priced products. In other words, for CR's low-end disruption scenario, the right-hand frame does not apply because there is no new market.

Each graph shows consumer demand[1] and product performance in relation to the given attribute. As illustrated in the left frame, low-end customers have lower demands for the first attribute than high-end customers. It is assumed users will buy a product only if it meets the user's demand along every dimension, but users will not pay extra for an overshoot in performance (i.e., they will buy the product that meets their demand by the smallest margin, with the least overshoot).[2] As illustrated in the left frame, the performance of the old product along this dimension meets the demand of both high-end and low-end consumers at time t_0. In contrast, the new product (the disruptive innovation) falls short of the low-end users' demand until time t_2, and falls short of the high-end customers' demand until time t_3 (we will address the demand curve for the new market shortly).

Consider the scenario CR and CAR call new-market disruption. Given our description above, this means the new market's demand curve for the first performance dimension must lie below that of the low-end market, as shown in the left frame of Figure B-1 (if it were positioned above the low-end market, then low-end users would buy the new product before the new-market users).[3] Now consider the right-hand frame in Figure B-1. CR show two demand curves for the second dimension – we interpret these to be the average demands for the old market and for the new market. Specifically, we interpret the lower curve to apply to the old market and the upper curve to the new market, such that the new market has a higher demand than the old market with regard to the second performance dimension. (Christensen (1992) shows that when a new smaller disk drive was introduced, it appealed to a new market which highly valued the new smaller size but placed lesser value on storage capacity, whereas the old market

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1 CR use the term "demand" to denote the performance level usable by the customer, without directly addressing the trade-off a customer generally makes between performance and price. We formalize this trade-off in our linear reservation price model.

2 This is not spelled out explicitly in C/CR/CAR, but it is implicit since the "demand curves" as developed in Christensen (1992) are simply the median disk drive capacities actually purchased by the various market segments. (For example, the high-end market is the mainframe segment.)

3 Technically, we should also consider the demands for the second performance dimension before drawing this conclusion – if we do so using the right-hand frame in Figure B-1, this statement still holds.

didn't place as much value on compactness. In other words, demands for the two performance dimensions were negatively correlated – the high end of the old market strongly demanded capacity but weakly demanded compactness, the low end of the old market less strongly demanded capacity and more strongly demanded compactness, while the new market least strongly demanded capacity and most strongly demanded compactness.) In the right frame of Figure B-1 we have drawn the demand curves in the relative positions suggested by these observations.[4]

Given these curves, the purchase decisions are as follows. Up to time t_2 the low-end and high-end users buy the old product (it exceeds both performance demands while the new product fails to meet demands for performance dimension 1). Up to time t_1 the new-market users buy nothing (neither product meets their performance demands for dimensions 1 or 2) while beyond time t_1 they buy the new product (it meets both demands with the least overshoot). The low-end users switch to the new product at time t_2 and the high-end users switch at time t_3. (Although both products meet their performance demands, the new product has less overshoot.)[5] Accordingly, we have shown that what C/CR/CAR call new-market disruption leads to low-end encroachment: the product first opens up a new market and then encroaches on the low-end market before diffusing up-market to the high end. In Schmidt and Porteus (2000) we illustrate how the new market can be on the low-end fringe of the old market (illustrating the "fringe-market" type) while in Druehl and Schmidt (2008) we formulate the "detached-market" type. That is, we have just shown that new-market disruption maps to low-end encroachment, which we further distinguish as being of either the fringe-market or detached-market type.

Our contention that disruption results in low-end encroachment has been derived above using the trajectory curves of C/CR/CAR. Since these trajectory curves form the basis of their theory of disruptive innovation, we conclude that C/CR/CAR effectively define disruptive innovation to be a low-end encroachment process, not a high-end one. The model presented in Druehl and Schmidt (2008), along with those in Schmidt and Porteus (2000) and in Schmidt and Druehl (2005), also support this notion.

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[4] The only possibility is that the "demand curve" for the users of the old product (i.e., the old market demand) lies below the performance curve for the old product (at least during the time period over which they buy the old product, or else they would not buy it). Similarly, the only possibility is that that the performance trajectory for the new product lies above the old market demand curve after the time at which the old market has switched to buying the new product.

[5] The trajectory for the new product in shown to start at t_0 but since no one buys until t_1 in this case we interpret this to mean the product is effectively introduced at time t_1.

To complete our mapping, we next address Christensen's categorizations of low-end disruption and sustaining innovation. Low-end disruption can be described as follows (since there is no new market in this case, we ignore the right frame in Figure B-1 and the new-market demand curve in the left frame). At time t_0 both the low-end and high-end users buy the old product, as the new product falls short of their demand. At time t_2 the low-end users switch to the new product as it catches up with their demand, and at time t_3 the high-end users switch. We have just described what we call low-end encroachment – the new product first sells to the low end and then diffuses upward to the high end. We call this the *immediate* form of low-end encroachment because there is no new market – the encroachment on the old market is immediate. CR use the example of discount retailers encroaching on high-end retailers. The analysis of Schmidt and Porteus (2000) also applies to the immediate form.

Mapping of Sustaining Innovation to High-end Encroachment

Next we discuss the diffusion pattern for a sustaining innovation. CAR's discussion of "undershot customers" suggests that sustaining innovations are repeatedly successful when customer demands are rapidly increasing relative to the rate of advancement in any given innovation. We use the familiar example of subsequent generations of Pentium microprocessors to illustrate this phenomenon. In the 1990s Intel would introduce a new generation of Pentium processor, and before long, because of further software upgrades by Microsoft, customers would be clamoring for more processing power (to date, the market for a new generation of MP3 player relative to the previous generation also fits this description, in that customers flock to a new model that stores more songs). We interpret this type of situation to mean that the users' demand curve is steep relative to the product's performance curve, as shown in Figure B-2. We again provide more detail in this trajectory chart than do C/CR/CAR, by inferring added detail from CAR's discussion of "undershot customers".[6]

[6] We interpret the trajectory chart on p. xvi in C as follows. After a disruptive innovation is introduced, such as a smaller disk drive, it is immediately and continually upgraded with an infinite number of infinitesimal sustaining innovations. This is what gives it the upward performance trajectory along the first performance dimension – this upward trajectory can only be achieved by sustaining innovations. This is why C's performance curve for the disruptive innovation is labeled "progress due to sustaining innovations." With regard to a sustaining innovation, there does not seem to be formal recognition of a discrete sustaining innovation (such as a new generation of microprocessor). Similar to what happens to a disruptive innovation after its introduction, the "non-disruptive" product (to which the trajectory chart on the left in C, p. xvi applies) simply undergoes an infinite number of infinitesimal sustaining innovations over time (the left trajectory is similarly labeled "progress due to sustaining innovations"). See also the trajectory chart on

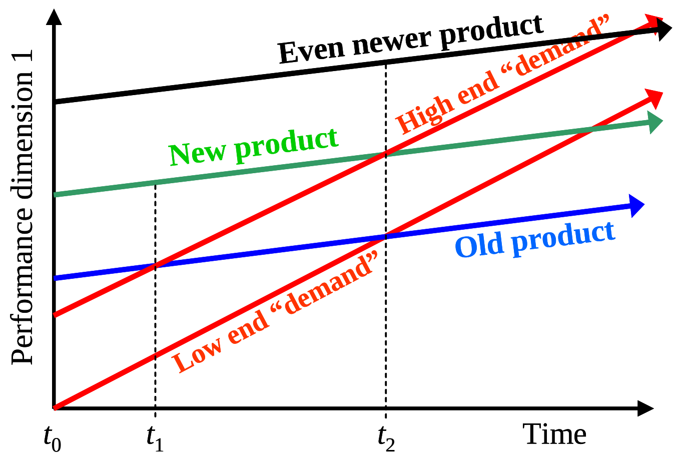

Figure B-2. Trajectory chart for a sustaining innovation

As suggested by Figure B-2, from time t_0 to t_1 both low-end and high-end users buy the old product, as it meets their demands with the least amount of overshoot. At time t_1 the high-end market switches to the new product as the old product no longer meets their needs (this implies the new product isn't introduced in the market until time t_1), while the low-end market does not switch to the new product until time t_2, at which point the high-end market moves on to an even newer product (another sustaining innovation, introduced, technically, at time t_2). Thus we infer sustaining innovation leads to high-end encroachment, where the new product diffuses downward from the high end of the existing market. Going back to the microprocessor example, a new generation of Pentium offers more of what current high-end customers want, processing power. It starts out high priced, selling first to high-end customers, diffusing downward.

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p. xvi in CAR with the curved arrows – in the context of the microprocessor we would interpret these to represent upgrades in a generation of microprocessor over time, such as from 500 to 600 to 700 MHz.

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APPENDIX C: Development of Opposite-Sloping Reservation Price Curves for Cell Phones

In this appendix we develop hypothetical part-worth curves for cell phones, to illustrate how they might result in opposite-sloping reservation price curves. (The new and old products are monthly cell-phone and land-line subscriptions, respectively.) At their introduction around twenty years ago, denoted by time t_0, cell phones were quite expensive and very bulky. Furthermore, reception was so poor that early users frequently lost contact with the party to whom they were speaking. But poor performance along this traditional dimension of quality (reception) was overlooked by early users because they more highly valued performance along an alternate dimension, which we call portability. In addition, phones can be thought of as conferring social status on the user. For example, in 1985, teens might have envied a peer who had her own individual land line. Similarly, an early cell-phone user might have purchased that phone in part because it called attention to her when the phone rang in public.

We assume reception, portability and status are the only features of significance for phones (cell or land line). Thus the number of attributes, denoted by n, is n = 3. We posit that various customers held (and continue to hold) vastly different willingness-to-pay for these attributes. For example, Rogers (2003) suggests early cell-phone users were business executives such as building contractors who highly valued phone portability in going from job to job. Such a user really represented multiple customers, because she still relied on a land line for basic service, with a uniquely different willingness-to-pay for a land line as compared to a cell phone.

We assume z consists of w distinct customer segments of size z_α, $\alpha \in \{1, ..., w\}$; for phones we assume w = 2, representing the stationary (α = 1) and mobile (α = 2) segments. Stationary customers use a phone in a relatively fixed location, such as in an office or a home. Mobile customers need a phone while on the go. Since from Figure 3 of Druehl and Schmidt (2008) we see that cell phones seem to encroach on the land-line market before their sales equal the sales of land lines, we assume $z_1 = 2\, z_2 = 2\, z/3$.

Starting with segment 1 and attribute 1, we order the customers from highest to lowest part worth, and assume the resulting part-worth curve is continuous and linear. This effectively identifies each customer's "type," denoted by $\theta_1 \in [0,\ z_1]$. Using this same ordering of customers within segment 1, we plot the part-worth curves for all other attributes and assume all part-worth curves are linear. We then proceed to each subsequent segment and again order customers from highest to lowest willingness-to-pay for some given attribute, such that customer type in segment α, $\alpha \in \{2, \ldots, w\}$, is denoted by $\theta_\alpha \in \left[\sum_{1}^{\alpha-1} z_\alpha, \sum_{1}^{\alpha} z_\alpha \right]$. Again, we assume the plot is continuous and linear, and assume the plots of the part-worth curves for all other attributes are linear within each customer segment.

For cell phones, $\theta_1 \in [0, 2z/3]$ and $\theta_2 \in [2z/3, z]$ denote stationary and mobile customers, respectively. We divide each segment into three sub-segments called business, individual, and teen. First consider the stationary customer segment. We assume that customers in the stationary business sub-segment are contemplating the purchase of an office (business) phone, stationary individual customers are considering a primary line for their homes, and teen customers are considering a second line for the home (other uses for a second line, such as for Internet access, will also fall under this sub-segment). First consider the attribute of reception. We assume business users have the highest part worths, followed by individuals, followed by teens. The logic is that an office user typically expects and demands a "perfect" connection. An individual home user expects this as well but is a bit more price sensitive. And a home user is typically willing to pay more for the first line than the second line, such that the part worth for the teen user (i.e., for the second home line) is lowest. These assumptions yield reception part-worth curves for stationary customers as shown in the left portion of Figure C-1. Customers are ordered from highest to lowest in terms of part worth for the reception attribute at time t_0, the time of introduction of the cell phone (roughly 1985). As suggested by the *x*-axis, business customers are of lowest type θ_1 followed by individuals followed by teens. We assume that z = 3,750,000 at t_0 and plot the part worths for land lines and cell phones.

We denote the slope of the part-worth curve for a particular time by $k_{i,\alpha}^{j}$ and the intercept by $v_{i,\alpha}^{j}$, where $j \in \{O, N\}$ denotes the product (the two products are identified as products *O* and *N*, where *O* is for old product and *N* is for new product, $i \in \{1, \ldots, n\}$ the attribute, and $\alpha \in \{1, \ldots, w\}$ the segment. Thus the part-worth (i.e., utility) curve is $u_{i,\alpha}^{j}(\theta_\alpha) = v_{i,\alpha}^{j} + k_{i,\alpha}^{j}\,\theta_\alpha$. We label reception as attribute 1, the land line as product *O*, and the cell phone as product *N*. We assume the land-line part-worth curve for reception for stationary customers in 1985 is $u_{1,1}^{O}(\theta_1) = v_{1,1}^{O} + k_{1,1}^{O}\,\theta_1 = 450 - 0.00013\,\theta_1$. (Note that these numbers represent hypothetical but plausible part worths.) All these customers consider the reception offered by a land line in 1985 to be worth more than that offered by a 1985 cell phone (product *N*). Hence, we assume $u_{1,1}^{N}(\theta_1) = 53 - 0.0000015\,\theta_1$.

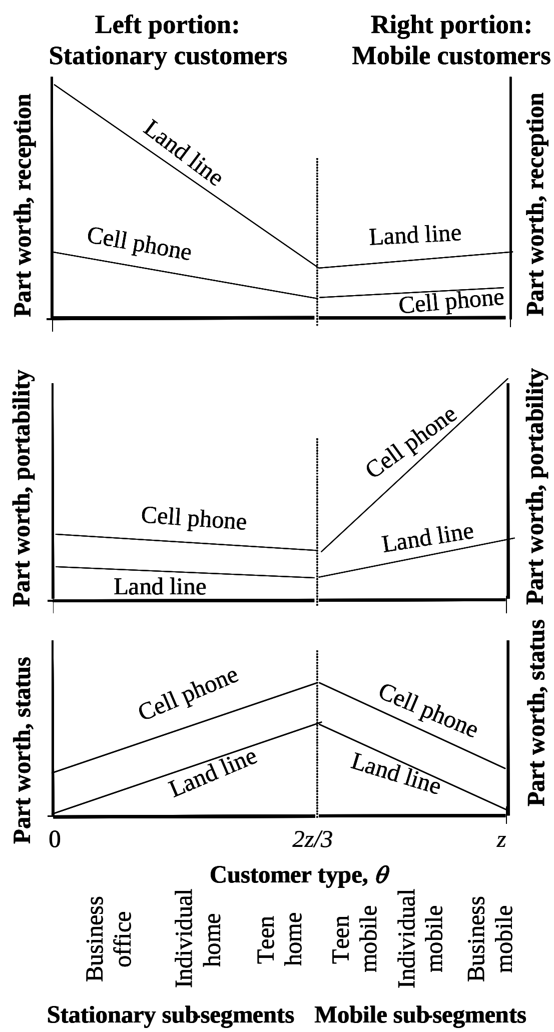

Figure C-1. Hypothetical part-worth curves for the phone example, circa 1985

Our framework thus accommodates the existence of multiple sub-segments assuming linear variation in willingness-to-pay within a sub-segment and that the sub-segments touch. For example, the "first" stationary teen customer is virtually identical to the "last" stationary individual customer.

Maintaining the same ordering of customers within the segment, we similarly plot the part-worth curves for all other attributes, and assume each part-worth curve is linear. To illustrate with phones, next consider the part worths that stationary customers ascribe to the attribute of portability, attribute 2 (see the left portion of the middle frame of Figure C-1), assuming the same ordering of customers as for reception. This attribute is not nearly as highly valued by stationary customers as reception, but part worths for business users are still higher than those of teen users because of the need for productivity, for example. We assume that for a land line the *y*-intercept is $v_{2,1}^{O} = 22$ and the slope is $k_{2,1}^{O} = -0.000001$. A cell phone clearly performs better than a land line along this dimension (a land line is portable only to the extent of the length of the phone cord, or the range of the base station), such that all customers hold a higher part worth for the cell phone. We therefore assume that $u_{2,1}^{N}\left(\theta_{1}\right)=43-0.0000012\,\theta_{1}$.

With regard to the part worth for status (the left side of the bottom frame in Figure C-1), we assume business customers are "all business" and accordingly do not place much value on this attribute, while individuals are more status conscious and teens are highly status conscious. Clearly, this is an oversimplification, but we proceed under this approximation. We assume the part worths are: $u_{3,1}^{O}\left(\theta_{1}\right)=45+0.00005\theta_{1}$ and $u_{3,1}^{N}\left(\theta_{1}\right)=66.5+0.0000754\,\theta_{1}$.

The mobile customer segment is illustrated in the right portion of Figure C-1. Without loss of generality, first order customers in terms of willingness-to-pay for the attribute of status. We again (simplistically) assume business customers are "all business" and teens are most swayed by status. We assume $u_{3,2}^{O}\left(\theta_{2}\right)\ =170-0.0001\theta_{2}$ and $u_{3,2}^{N}\left(\theta_{2}\right)\ =255-0.0001424\,\theta_{2}$, suggesting all customers attribute higher status to a cell phone. With regard to the attribute of reception, mobile customers were probably not so concerned with reception back in 1985, as shown in the right side of the top frame, but everyone would have rated a land line more favorably along this dimension than a cell phone, with mobile business customers willing to pay more than teens (and individuals in between). Thus we assume $u_{1,2}^{O}\left(\theta_{2}\right)\ =125+0.000008\,\theta_{2}$ and $u_{1,2}^{N}\left(\theta_{2}\right)\ =60.4+0.000009\,\theta_{2}$. However, as shown in the right portion of the middle frame, mobile customers highly value portability, and a cell phone's portability is much superior to that of a land line. Thus the slope of the cell-phone part worth is greater; we assume $u_{2,2}^{O}\left(\theta_{2}\right)=19.5+0.000011\theta_{2}$ and $u_{2,2}^{N}\left(\theta_{2}\right)\ =40+0.0002061\theta_{2}$.

Note that as is the case with a disruptive innovation, phone customers are assumed to have differed significantly with regard to the value they placed on individual product attributes. Stationary customers highly valued reception, while mobile customers highly valued portability. The new (disruptive) cell phone was weak on the first dimension, but strong on the second.

To obtain the reservation price curves for each customer segment, we sum the part worths within that segment. Define $k_{\alpha}^{j}:=\sum_{i=1}^{n}k_{i,\,\alpha}^{j}$ and $v_{\alpha}^{j}:=\sum_{i=1}^{n}v_{i,\,\alpha}^{j}$, such that $u_{\alpha}^{j}\left(\theta_{\alpha}\right):=\sum_{i=1}^{n}u_{i,\,\alpha}^{j}\left(\theta_{\alpha}\right)=v_{\alpha}^{j}+\theta_{\alpha}\ k_{\alpha}^{j}$ denotes the customer's reservation price for product *j* within segment α. Thus for our phone example, $u_{1}^{O}\left(\theta_{1}\right)$ $=517-0.000081\theta_{1}$ and $u_{2}^{O}\left(\theta_{2}\right)\ =\ 315-0.000081\ \theta_{2}$, such that $k_{1}^{O}=k_{2}^{O}:=k_{O}\ =\ -0.000081$. Generalizing, our model assumes that for each product, the sums of the slopes of the part-worth curves are equal across all customer segments. (If for any given segment the sum of the slopes k_α^j is of opposite sign as compared to a different segment, reverse the order of customers within that segment.) We denote this sum by k_j for $j \in \{O, N\}$.

We further assume there is an ordering of customer segments such that $u_\alpha^j\left(\sum_1^\alpha z_\alpha\right) = u_{\alpha+1}^j\left(\sum_1^\alpha z_\alpha\right)$ for $\alpha \in \{1, \ldots, w-1\}$. Effectively, we assume the reservation price curves for the individual segments can be pieced together to form a linear reservation price curve for the product. (The segments may need to be reordered to achieve this.) Thus a customer of type θ_α can simply be referred to as being of type θ, where $\theta \in [0, z]$, effectively yielding a uniform distribution of customer types over the interval $[0, z]$. (Technically, uniformity will not hold at z_1, $z_1 + z_2$, …, $z_1 + ... + z_{w-1}$, but we ignore this as the effect is infinitesimal.) The reservation price for a customer of type θ is simply denoted by $u_j(\theta)$ and the reservation price curve for product *j* can thus be described by $u_j(\theta) = v_j + \theta\, k_j$ where $v_j := u_1^j(0)$ $= \sum_{i=1}^n v_{i,\,1}^j$.

Using the most recently defined ordering of *w* customer segments and ordering of customers within each segment, the part worths for the other product *J* are plotted, $J \in \{O, N\}$ and $J \neq j$. Again it is assumed that the sums of the slopes are equal, $k_J := \sum_{i=1}^n k_{i,\,1}^J = \ldots = \sum_{i=1}^n k_{i,\,w}^J$, but k_J need not equal k_j, and that the linear reservation price curves for the various segments can be pieced together to form a linear reservation price curve for the product. Recall that without loss of generality k_j is negative; in Druehl and Schmidt (2008) we provide analytical results for the case where k_J is positive.

Given the above setup, product *J*'s reservation price curve can be described by $u_J(\theta) =$ $v_J + \theta\, k_J$ where $v_J := u_1^J(0) = \sum_{i=1}^n v_{i,\,1}^J$. For phones, the stationary and mobile segments meet at $2z/3$, where they are joined. The resulting reservation price curve for land lines is $u_O(\theta) = 517 - 0.000081\theta$, while for cell phones it is $u_N(\theta) = 162.5 + 0.0000727\,\theta$.

For ease of exposition we normalize the curves such that $v_O = 1$, $k_O = -1$, and $z = 0.5$. This results in the reservation price curves in Druehl and Schmidt (2008), $u_O(\theta) = 1 - \theta$ and $u_N(\theta) = 0.314 + 0.9\ \theta$.

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