Appendix

PROOF to PROPOSITION 3.1.

Let $\dot{\widetilde{m}}(t) = -\varepsilon\widetilde{m}(t)$, $\widetilde{m}(0) = m^0 \geqslant 0$. Then $\widetilde{m}(t) = m^0 e^{-\varepsilon t} \geqslant 0$, for $0 \leqslant t \leqslant T$. Since $ah(P_2, m, Q_2)g(0, Q_1) + h(P_2, m, Q_2) \geqslant 0$, it is obvious that $m(t) \geqslant \widetilde{m}(t) \geqslant 0$, where $m(t)$ is a solution of (6) for arbitrary $P_2(t)$ and $Q_1(t)$. Since $m(t) \geqslant 0$, we can similarly conclude that $Q_1(t) \geqslant 0$ for $0 \leqslant t \leqslant T$. Moreover, it is easy to see that $m^0 > 0$ and $Q_1^0 > 0$ imply $m(t) > 0$ and $Q_1(t) > 0$ for $0 \leqslant t \leqslant T$. □

PROOF to PROPOSITION 3.2.

(a) From (4), we see that $V_O^\alpha(0, Q_1^0, m^0) = J(P_2^*(t \mid \alpha))$ or simply $V_O^\alpha = J(P_2^*(t \mid \alpha))$, where $P_2^*(t \mid \alpha)$ is the optimal price trajectory given α. Let $Q_1^{P_2(t)}(t \mid \alpha)$ denote the software quality trajectory given a price trajectory $P_2(t)$ and α. Similarly, let $m^{P_2(t)}(t \mid \alpha)$ denote the user network size trajectory given a price trajectory $P_2(t)$ and α.

From (5), we have $Q_1(t) = Q_1^0 e^{-\delta t} + \alpha e^{-\delta t}\int_0^t e^{\delta\tau} m(\tau)d\tau$, which increases with α for every fixed trajectory $m(t)$. Next we see from (6) that for a given price trajectory $P_2(t) \geqslant 0$, $\frac{\partial \dot{m}(t)}{\partial Q_1(t)} = ah(P_2(t), m(t), Q_2(t))\frac{\partial g(t)}{\partial Q_1(t)} \geqslant 0$, which means that $\dot{m}(t)$ increases with $Q_1(t)$. This implies that both $Q_1(t)$ and $m(t)$ increase as α increases for a given price trajectory $P_2(t) \geqslant 0$.

Let $0 < \alpha_1 < \alpha_2$. Then $Q_1^{P_2(t)}(t \mid \alpha_1) \leqslant Q_1^{P_2(t)}(t \mid \alpha_2)$ and $m^{P_2(t)}(t \mid \alpha_1) \leqslant m^{P_2(t)}(t \mid \alpha_2)$. By the assumption on the functions h and σ, it is apparent that $J(P_2(t \mid \alpha_1) \leqslant J(P_2(t \mid \alpha_2)$. By definition, $J(P_2(t \mid \alpha_1) \leqslant J(P_2^*(t \mid \alpha_1) \leqslant J(P_2(t \mid \alpha_2) \leqslant J(P_2^*(t \mid \alpha_2)$. Therefore, $V_O^{\alpha_1} \leqslant V_O^{\alpha_2}$. This completes the proof.

(b) The proof is similar to part (a). □

PROOF to PROPOSITION 3.3.

From Proposition 3.2, part (b), we know that $\frac{\partial V_O(0, Q_1(0))}{\partial Q_1(0)} \geqslant 0$. Therefore, $\lambda(0) = \frac{\partial V_O(0, Q_1(0))}{\partial Q_1(0)} \geqslant 0$. The same argument extends to $\lambda(t) = \frac{\partial V_O(0, Q_1(t))}{\partial Q_1(t)} \geqslant 0$. □

PROOF to PROPOSITION 3.4. This proof requires Lemma .1.

LEMMA .1 *In the open source model, $P_2 + \mu(ag(0, Q_1) + 1) \geqslant 0$, for $0 \leqslant t \leqslant T$.*

PROOF to LEMMA .1.

According to (11) and (12), we know that there are two cases:

Case1:

$$\left\{h(P_2, m, Q_2) + [P_2 + \mu(ag(0, Q_1) + 1)]\,\frac{\partial h}{\partial P_2}\right\}\Big|_{P_2=0} \leqslant 0$$

and $\eta_2 \geqslant 0$,

and Case2:

$$\left\{h(P_2, m, Q_2) + [P_2 + \mu(ag(0, Q_1) + 1)]\,\frac{\partial h}{\partial P_2}\right\}\Big|_{P_2>0} = 0$$

and $\eta_2 = 0$.

In case 1, $[P_2 + \mu(ag(0, Q_1) + 1)]|_{P_2=0} = \mu(ag(0, Q_1) + 1) \geqslant -h(P_2, m, Q_2)/\frac{\partial h}{\partial P_2}\Big|_{P_2=0} \geqslant 0$. In case 2, $[P_2 + \mu(ag(0, Q_1) + 1)]|_{P_2>0} = -h(P_2, m, Q_2)/\frac{\partial h}{\partial P_2}\Big|_{P_2>0} \geqslant 0$. The result follows.

By contradiction. Suppose at an arbitrarily chosen time $\tau \in [0, T]$, $\mu(\tau) < 0$. By Proposition 3.3 and Lemma .1,

$$\dot{\mu} = (\rho + \varepsilon)\mu - \alpha\lambda - [P_2 + \mu(ag(0, Q_1) + 1)]\,\frac{\partial h}{\partial m} < 0.$$

Therefore, $\mu(\tau) < 0$ for $\tau \leqslant t \leqslant T$. This contradicts $\mu(T) \geqslant 0$. So $\mu(\tau) \geqslant 0$. Since τ is arbitrary, we can conclude that $\mu(T) \geqslant 0$ for $0 \leqslant t \leqslant T$. □

PROOF to PROPOSITION 3.5.

If $P_2^* > 0$, then $h + P_2\frac{\partial h}{\partial P_2} + \mu(ag(0, Q_1) + 1)\frac{\partial h}{\partial P_2}\Big|_{P_2^*} = 0$ (from (11)). By Proposition 3.4 and the assumptions that $g \geqslant 0$ and $\frac{\partial h}{\partial P_2} \leqslant 0$, we have $\mu(ag(0, Q_1) + 1)\frac{\partial h}{\partial P_2}\Big|_{P_2^*} \leqslant 0$. Therefore, $h + P_2\frac{\partial h}{\partial P_2}\Big|_{P_2^*} \geqslant 0$. By definition, $\frac{\partial F_O}{\partial P_2}\Big|_{\hat{P}_2} = h + P_2\frac{\partial h}{\partial P_2}\Big|_{\hat{P}_2} = 0$. Then $\hat{P}_2 = h + P_2\frac{\partial h}{\partial P_2}\Big|_{\hat{P}_2} \geqslant 0$. By the concavity of F_O, we conclude $P_2^*(t) \leqslant \hat{P}_2(m^*(t), Q_2(t))$ for $0 \leqslant t \leqslant T$. Moreover, if the salvage value is zero at time T, then $\mu(T) = 0$. From the previous argument, it is easy to show $P_2^*(T) = \hat{P}_2(m^*(T), Q_2(T))$. □

PROOF to PROPOSITION 3.6. The proof is similar to that of Proposition 3.1. □

PROOF to PROPOSITION 3.7.

The proofs for part (a) and (b) are similar to that of Proposition 3.2. (c) Using the Envelope Theorem (e.g., Varian, 1978, Page 268), we have $\frac{dV_C}{dw} = \frac{\partial L}{\partial w} = -\int_0^T N^2 dt$. Therefore, the optimal closed source profit decreases with w. □

PROOF to PROPOSITION 3.8. The proof is similar to that of Proposition 3.3. □

PROOF to PROPOSITION 3.9 .

The proof requires Lemma .2 and Proposition 3.8.

LEMMA .2 *In the closed source model,* $P_1 + a\mu \geqslant 0$ *and* $P_2 + \mu + (P_1 + a\mu)g(P_1, Q_1) \geqslant 0$, *for* $0 \leqslant t \leqslant T$.

PROOF to LEMMA .2. The proof is similar to that of Lemma .1. □

By contradiction. Suppose at an arbitrarily chosen time $\tau \in [0, T]$, $\mu(\tau) < 0$. By Proposition 3.8 and Lemma .2,

$$\dot{\mu} = (\rho + \varepsilon)\mu - [P_2 + \mu + (P_1 + a\mu)g(P_1, Q_1)] \frac{\partial h}{\partial m} < 0.$$

Therefore, $\mu(\tau) < 0$ for $\tau \leqslant t \leqslant T$. This contradicts $\mu(T) \geqslant 0$. So $\mu(\tau) \geqslant 0$. Since τ is arbitrary, we can conclude that $\mu(T) \geqslant 0$ for $0 \leqslant t \leqslant T$. □

PROOF to PROPOSITION 3.10.

If $P_1^* > 0$, then $\left(g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1}\right) h(P_2, m, Q_2) + a\mu \frac{\partial g}{\partial P_1} h(P_2, m, Q_2)\Big|_{P_1^*, P_2^*} = 0$ (from (20)). We can also say that $g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1} + a\mu \frac{\partial g}{\partial P_1}\Big|_{P_1^*} = 0$. By Proposition 3.9 and the assumption that $\frac{\partial g}{\partial P_1} \leqslant 0$, we have $a\mu \frac{\partial g}{\partial P_1}\Big|_{P_1^*} \leqslant 0$. Therefore, $g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1}\Big|_{P_1^*} \geqslant 0$. By definition, $\frac{\partial F_C}{\partial P_1}\Big|_{\hat{P}_1, \hat{P}_2} = \left(g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1}\right) h(P_2, m, Q_2)|_{\hat{P}_1, \hat{P}_2} = 0$. We can also say that $g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1}\Big|_{\hat{P}_1} = 0$. Then $\hat{P}_1 = -g / \frac{\partial g}{\partial P_1}\Big|_{\hat{P}_1} \geqslant 0$. By the concavity of F_C, we conclude $P_1^*(t) \leqslant \hat{P}_1(m^*(t), Q_1^*(t), Q_2(t))$ for $0 \leqslant t \leqslant T$. Similarly, if $P_2^* > 0$, then $h(P_2, m, Q_2) + [P_2 + P_1 g(P_1, Q_1)] \frac{\partial h}{\partial P_2} + \mu(ag(P_1, Q_1) + 1) \frac{\partial h}{\partial P_2}\Big|_{P_1^*, P_2^*} = 0$ (from (21)). Clearly, P_2^* is a function of P_1^*. Let $P_2^*(P_1)$ is the solution to $h(P_2, m, Q_2) + [P_2 + P_1 g(P_1, Q_1)] \frac{\partial h}{\partial P_2} + \mu(ag(P_1, Q_1) + 1) \frac{\partial h}{\partial P_2} = 0$. It can be shown that $P_2^*(P_1^*) \leqslant P_2^*(\hat{P}_1)$. By Proposition 3.9 and the assumptions that $g \geqslant 0$ and $\frac{\partial h}{\partial P_2} \leqslant 0$, we have $\mu(ag(P_1, Q_1) + 1) \frac{\partial h}{\partial P_2}\Big|_{P_1^*, P_2^*} \leqslant 0$. Therefore, $h(P_2, m, Q_2) + [P_2 + P_1 g(P_1, Q_1)] \frac{\partial h}{\partial P_2}\Big|_{P_1^*, P_2^*} \geqslant 0$. By definition, $\frac{\partial F_C}{\partial P_2}\Big|_{\hat{P}_2} = h(P_2, m, Q_2) + [P_2 + P_1 g(P_1, Q_1)] \frac{\partial h}{\partial P_2}\Big|_{\hat{P}_1, \hat{P}_2} = 0$. Clearly, $\hat{P}_2$ is a function of $\hat{P}_1$. We denote it as $\hat{P}_2(\hat{P}_1)$. By the concavity of F_C, we know that $P_2^*(\hat{P}_1) \leqslant \hat{P}_2(\hat{P}_1)$. Therefore, $P_2^*(P_1^*) \leqslant \hat{P}_2(\hat{P}_1)$. we conclude $P_2^*(t) \leqslant \hat{P}_2(m^*(t), Q_1^*(t), Q_2(t))$ for $0 \leqslant t \leqslant T$. Moreover, if the salvage value is zero at time T, then $\mu(T) = 0$. From the previous argument, it is easy to show $P_1^*(T) = \hat{P}_1(m^*(T), Q_1^*(t), Q_2(T))$ and $P_2^*(T) = \hat{P}_2(m^*(T), Q_1^*(t), Q_2(T))$. □

PROOF to COROLLARY 4.1.

(i) Exponential demand function. From Proposition 4.1, $P_2^*(t) \leqslant m^*(t)Q_2(t)$. From (6), $\dot{m} = (a\exp(-\frac{c}{Q_1}) + 1)\exp(-\frac{P_2}{mQ_2}) - \varepsilon m, m(0) = m^0$. Let $\dot{\bar{m}} = (a + 1) - \varepsilon m, \bar{m}(0) = m^0$. Then $\bar{m}(t) = m^0 e^{-\varepsilon t} + (a + 1)(1 - e^{-\varepsilon t})/\varepsilon$, $0 \leqslant t \leqslant T$. Clearly, $\dot{\bar{m}} > \dot{m}$. The result follows.

(ii) Linear-price demand function. Proof is similar to (i). □

PROOF to COROLLARY 4.2.

(i) Exponential demand function. From Proposition 4.1, $P_1^*(t) \leqslant Q_1^*(t)$. From (14), $\dot{Q}_1 = kN - \delta Q_1, Q_1(0) = Q_1^0$. Let $\dot{\bar{Q}}_1 = kN(0) - \delta\bar{Q}_1, \bar{Q}_1(0) = Q_1^0$, where $N(0)$ is the number of in-house programmers at time 0. Then $\bar{Q}_1(t) = Q_1^0 e^{-\delta t} + kN(0)(1 - e^{-\delta t})/\delta$, $0 \leqslant t \leqslant T$. $\dot{\bar{Q}}_1 > \dot{Q}_1$ since N is decreasing over time. The result follows.

From Proposition 4.1, $P_2^*(t) \leqslant m^*(t)Q_2(t) - \hat{P}_1^*(t)g(\hat{P}_1^*(t), Q_1^*(t)) \leqslant m^*(t)Q_2(t)$. From (15), $\dot{m} =$

$(a\exp(-\frac{P_1+c}{Q_1})+1)\exp(-\frac{P_2}{mQ_2})-\varepsilon m, m(0) = m^0$. Let $\dot{\bar{m}} = (a+1)-\varepsilon m, \bar{m}(0) = m^0$. Then $\bar{m}(t) = m^0 e^{-\varepsilon t}+(a+1)(1-e^{-\varepsilon t})/\varepsilon$, $0 \leqslant t \leqslant T$. Clearly, $\dot{\bar{m}} > \dot{m}$. The result follows.

(ii) Linear-price demand function. Proof is similar to (i). □