Electronic Companion to "Scheduling Support Times for Satellites with Overlapping Visibilities"

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Appendix A: Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1: Without loss of generality, we let $a = 0$ and $b = L$. The objective function corresponding to a support-time x_i for Satellite $S_i, i = 1, 2, ..., m$, is

$$\text{Max } \sum_{i=1}^{m} \Pi_i(x_i) = d_1(1 - e^{-\beta_1 x_1}) + d_2(1 - e^{-\beta_2 x_2}) + \ldots + d_m(1 - e^{-\beta_m x_m}) \tag{8}$$

Since the visibility time-windows for all the satellites are identical, the only restriction on the support-times x_i is

$$x_1 + x_2 + \ldots + x_m = L \tag{9}$$

Associating a multiplier λ with (9), we form the Lagrangian

$$\Upsilon(x_1, x_2, \ldots, x_m, \lambda) = \sum_{i=1}^{m} d_i(1 - e^{-\beta_i x_i}) + \lambda(L - (x_1 + x_2 + \ldots + x_m)) \tag{10}$$

It is easy to see that the first-order optimality conditions: $\partial\Upsilon/\partial x_i = 0; \partial\Upsilon/\partial\lambda = 0$ are both necessary and sufficient; the sufficiency follows from the fact that the Hessian matrix (which is diagonal in this case) of the Lagrangian is negative definite (see, e.g., Bazaraa et al. 1993). If $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m, \overline{\lambda})$ is the point maximizing the Lagrangian Υ, then we have

$$d_i \beta_i e^{-\beta_i \overline{x}_i} - \overline{\lambda} = 0, \; i = 1, 2, \ldots, m, \tag{11}$$

$$\overline{x}_1 + \overline{x}_2 + \ldots + \overline{x}_m = L \tag{12}$$

From (11), we have $\overline{x}_i = -\frac{1}{\beta_i} \ln(\frac{\overline{\lambda}}{\beta_i d_i}), \; i = 1, 2, \ldots, m$. Substituting these into (12), we get $\overline{\lambda}$. ∎

Proof of Theorem 1: Consider an arbitrary instance of 3-Partition (Garey and Johnson, 1979).

3-Partition: Given $B \in Z^+$, a set $A = \{z_1, z_2, \ldots, z_{3u}\}, z_i \in Z^+, i = 1, 2, \ldots, 3u$ with $B/4 < z_i < B/2$, and $\sum_{z_i \in A} z_i = uB$, does there exist a partition of A into disjoint subsets $A_1, A_2, \ldots, A_u$ such that (i) $|A_j| = 3$, and (ii) $\sum_{z_i \in A_j} z_i = B$ for $1 \le j \le u$?

Given an instance of 3-Partition, we construct a specific instance of the decision problem for Problem P_1 as follows:

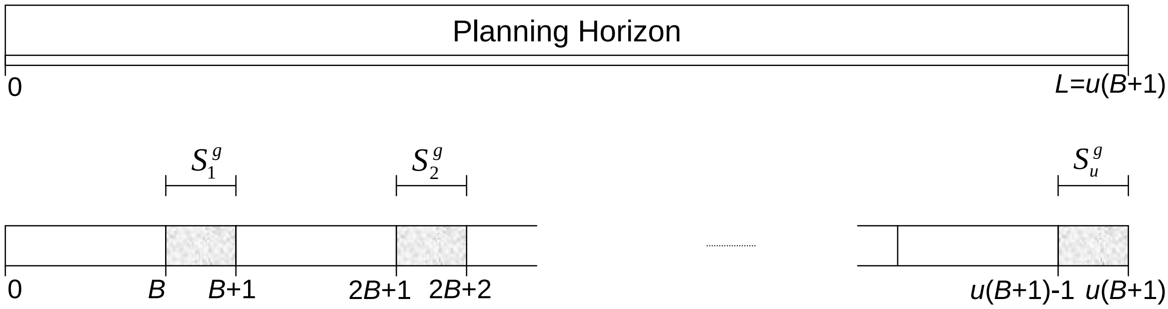

Figure 8: An Instance of the Decision Problem Q_1

- Consider a set of $4u$ satellites $M = \{S_i^z | 1 \le i \le 3u \cup S_i^g | 1 \le i \le u\}$. For ease of exposition, we let $S^z = \{S_i^z | 1 \le i \le 3u\}$ and $S^g = \{S_i^g | 1 \le i \le u\}$. Thus, $M = S^z \cup S^g$.

- The upper bound of the planning horizon, L, equals $u(B+1)$.

- The time-window of Satellite $S_i^g, i = 1, 2, \ldots, u$, is $[a_i^g, b_i^g] = [i(B+1) - 1, i(B+1)]$.

- The time-window of Satellite $S_i^z, i = 1, 2, \ldots, 3u$, is $[a_i^z, b_i^z] = [0, L]$.

- The utility functions $\Pi_i^g(x_i)$ (respectively $\Pi_i^z(x_i)$; see Figure 9) of Satellites S_i^g (resp. S_i^z), for $i = 1, 2, \ldots, u$ (resp. $i = 1, 2, \ldots, 3u$), are as defined below.

  1. $\Pi_i^g(x_i)$ and $\Pi_i^z(x_i)$ are differentiable and strictly concave (i.e., strictly increasing functions of the support-time x_i and $\Pi_i'^g(x_i)$ and $\Pi_i'^z(x_i)$ are strictly decreasing functions of x_i).

  2. $\Pi_i^z(z_i) = B + z_i$, $i = 1, 2, \ldots, 3u$.

  3. $\Pi_i'^z(z_i) = \Pi_j'^z(z_j)$, $1 \le i, j \le 3u$, where $\Pi_i'^z(z_i)$ is the derivative of the utility function of Satellite S_i^z at the support-time z_i.

4. $\Pi_i^g(1) = 3B$, $i = 1, 2, \ldots, u$.

It is easy to see that the set of functions that satisfy the properties above is nonempty. For example, the functional forms $\Pi_i^g(x_i) = d_i^g(1 - e^{-\beta_i^g x_i})$ and $\Pi_i^z(x_i) = d_i^z(1 - e^{-\beta_i^z x_i})$ satisfy these properties (see Appendix B).

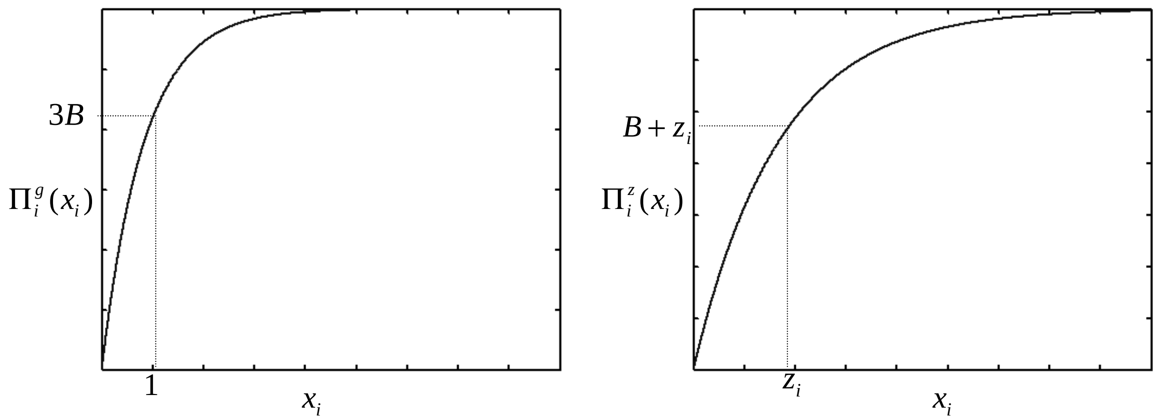

Figure 9: Utility Functions for Satellites S_i^g and S_i^z

For this specific instance of Problem P_1, consider the following decision problem (see Figure 8).

<u>DECISION PROBLEM (Q_1)</u>: Does there exist a schedule σ of support-times such that the total utility over the planning horizon $[0, L]$ is at least $7uB$?

The decision problem Q_1 is clearly in class NP. Also, it is easy to verify that the construction of Q_1 can be done in polynomial-time. We now show that there is a schedule σ of support-times such that $\Pi_\sigma \geq 7uB$ if and only if there exists a solution to the 3-Partition problem.

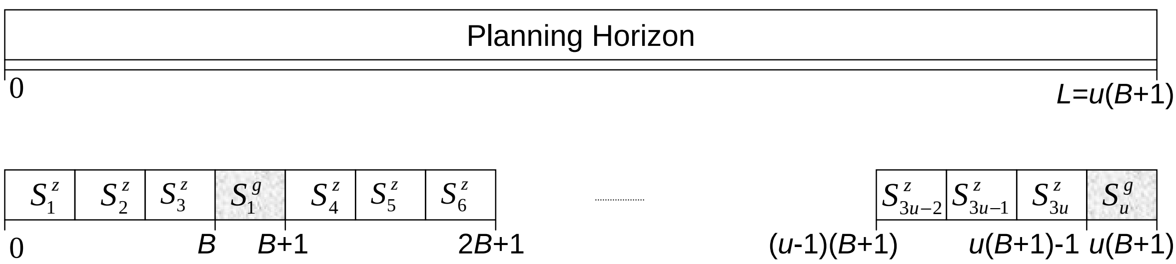

Figure 10: The Proposed Schedule σ

If part: If there exists a 3-Partition, then there is a schedule σ of support-times with a total utility of $\Pi_\sigma = 7uB$. The proposed schedule σ is illustrated in Figure 10. S_i^g is the only satellite supported in the time slot $[i(B+1)-1, i(B+1)]$, $i = 1, 2, \ldots, u$. We schedule exactly three satellites from S^z in each time slot $[i(B+1)-(B+1), i(B+1)-1]$, $i = 1, 2, \ldots, u$, having support-times z_{3i-2}, z_{3i-1}, z_{3i} corresponding to 3-Partition elements with $z_{3i-2} + z_{3i-1} + z_{3i} = B$, $i = 1, 2, \ldots, u$. Thus, there is no idle-time in the schedule σ; the total utility is $\Pi_\sigma = 7uB$.

Only if part: If there exists a support schedule σ_0 with $\Pi_{\sigma_0} \geq 7uB$, then we prove the existence of a 3-Partition through a series of claims. Consider the following problem.

Problem $\mathbf{R_1}$: Satellites S_i^z, $i = 1, 2, \ldots, 3u$ are all visible during the interval $[0, L']$, $L' = uB$. As before, there is no reconfiguration time. Find an optimum non-preemptive solution which maximizes the total utility.

Claim 1. The optimum solution value (i.e., utility) of Problem R_1 is $4uB$ and the unique corresponding schedule is one in which Satellite S_i^z has a support-time of z_i, $i = 1, 2, \ldots, 3u$.

Proof: Note that $(z_1, z_2, \ldots, z_{3u})$ is a feasible solution to Problem R_1 as $\sum_{i=1}^{3u} z_i = uB$. Moreover, for this solution, the individual utility functions have the same slope, i.e., $\Pi'^z_i(z_i) = \Pi'^z_j(z_j)$, $1 \leq i, j \leq 3u$ and the total utility is $\sum_{i=1}^{3u}(B + z_i) = 4uB$. It remains to be shown that the solution $(z_1, z_2, \ldots, z_{3u})$ is optimal.

Consider an optimum solution $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{3u}) \neq (z_1, z_2, \ldots, z_{3u})$. Since the utility functions are strictly increasing functions of the support-times, we may assume that $\sum_{i=1}^{3u} \hat{z}_i = uB$. There exists two support times $z_j < \hat{z}_j$ and $z_k > \hat{z}_k$. Then, we have $\Pi'_j(\hat{z}_j) < \Pi'_j(z_j) = \Pi'_k(z_k) < \Pi'_k(\hat{z}_k)$, which contradicts the fact that the marginal utilities must be equal at the optimum. Thus, the uniqueness of the optimum solution follows from the strict concavity of the utility functions.

All the following claims prove relevant properties for the required support schedule, σ_0, of Problem Q_1.

Claim 2. In schedule σ_0, it is necessary for a satellite from S^g to be supported during the time slot $[i(B+1)-1), i(B+1)]$, $i = 1, 2, \ldots, u$.

Proof: The utility of providing one time unit of support for Satellite S_i^g $(3B)$ is much larger than providing the same support for Satellite S_i^z (see Figure 9). Suppose only $(u-1)$ satellites from S^g are provided support of one unit time each. Then, the total utility for this scenario is at most $3(u-1)B + 4uB + (B + z_i)$. The first term is the contribution to the utility of the $(u-1)$ satellites of S^g. The second term, due to Claim 1, is the maximum contribution that can be obtained from the satellites in S^z. The third term is the upper bound on the maximum utility for a satellite in S^z for one unit of time. Since $3(u-1)B + 4uB + (B + z_i) < 7uB$, all the satellites in S^g, $i = 1, 2, \ldots, u$, must be supported during the entire duration that they are available.

It follows from the above claim that satellites in S^z can only be supported in the time slots $[i(B+1) - (B+1), i(B+1) - 1]$, $i = 1, 2, \ldots, u$. Suppose each satellite S_i^z is supported for an amount of time, $\hat{z}_i$, where $\hat{z}_i \geq 0$. Since the utility function is a continuous and increasing function of time, the constraint on the sum of support-times is $\hat{z}_1 + \hat{z}_2 + \ldots + \hat{z}_{3u-1} + \hat{z}_{3u} = uB$.

<u>Claim 3</u>. The total contribution to the utility of the Satellites S_i^z, $i = 1, 2, \ldots, 3u$ must be at least $4uB$.

Proof: Since the satellites in S^g contribute a total utility of $3uB$, the utility contribution of satellites in S^z, $i = 1, 2, \ldots, 3u$ must be at least $4uB$.

Based on Claims 1 and 3, we infer that $\hat{z}_i = z_i$, $i = 1, 2, \ldots, 3u$, and $z_1 + z_2 + \ldots + z_{3u-1} + z_{3u} = uB$. Thus, the total utility, Π_{σ_0}, equals $7uB$.

<u>Claim 4</u>. There must be exactly three satellites in S_i^z, $i = 1, 2, \ldots, 3u$, that are supported in each time slot $[i(B+1) - (B+1), i(B+1) - 1]$, $i = 1, 2, \ldots, u$.

Proof: From the discussion above, it follows that each satellite in S_i^z is supported for a duration z_i, and $z_1 + z_2 + \ldots + z_{3u-1} + z_{3u} = uB$. Thus, there is no idle-time in the required schedule σ_0. If only two satellites were served in the time slot, $[i(B+1) - (B+1), i(B+1) - 1], i = 1, 2, \ldots, u$, then there would be an idle-time in this slot as $z_i < B/2$. Four satellites cannot be served in any time slot $[i(B+1) - (B+1), i(B+1) - 1], i = 1, 2, \ldots, u$, since $z_i > B/4$. Thus, exactly three satellites whose total duration equals B are supported in each time slot $[i(B+1) - (B+1), i(B+1) - 1], i = 1, 2, \ldots, u$. Therefore, there exists a 3-partition of A into disjoint subsets $A_1, A_2, \ldots, A_u$ such that, for $1 \leq j \leq u$, $\sum_{z_i \in A_j} z_i = B$ and $|A_j| = 3$. ■

Appendix B: Existence of the Functions Required in the Proof of Theorem 1

Given positive constants B and $z_i, i = 1, ..., 3u$ (specified by an arbitrary instance of 3-partition), the construction in Theorem 1 requires the existence of functions $\Pi_i^z : x_i \rightarrow \Re_+, i = 1, ..., 3u$, that satisfy the following properties:

1. $\Pi_i^z, i = 1, ..., 3u$, are differentiable, increasing functions of the support-time x_i.
2. $\Pi_i^z(z_i) = B + z_i$, $i = 1, 2, \ldots, 3u$.
3. $\Pi'^z_i(z_i) = \Pi'^z_j(z_j)$, $1 \leq i, j \leq 3u$.

We show below, via a constructive proof, that there exist functions of the form $\Pi_i^z(x_i) = d_i(1 - e^{-\beta_i x_i})$ that satisfy the properties above. For this functional form, Property (3) becomes

$$d_i \beta_i e^{-\beta_i z_i} = d_j \beta_j e^{-\beta_j z_j}, 1 \leq i, j \leq 3u$$

It is sufficient to explain the construction for two such functions, say Π_1^z and Π_2^z; the construction can be repeated to obtain the other functions $\Pi_i^z, i = 3, 4, ..., 3u$.

Define $d_1 = \frac{B+z_1}{1-e^{-\beta_1 z_1}}$. Let $\beta_1 > 0$ be such that $C = d_1 \beta_1 e^{-\beta_1 z_1} = \frac{\beta_1 e^{-\beta_1 z_1}}{1-e^{-\beta_1 z_1}}(B + z_1) \leq 3$. This is possible since $\frac{\beta_1 e^{-\beta_1 z_1}}{1-e^{-\beta_1 z_1}}$ is a decreasing function of β; note that $\beta_1 = O(\frac{\log(B+z_1)}{z_1})$ suffices for our purpose. Note that Π_1^z satisfies Properties (1) and (2). The reason for bounding C from above is the following.

CLAIM: $z_i C < B + z_i, i = 2, 3, ...3u$.

Proof: Note that $2z_i < B$ (refer to the proof of Theorem 1 in Appendix A). Thus, $z_i C \leq 3z_i < B + z_i$. ∎

It follows from the claim above that there exists $\beta_2 > 0$ satisfying

$$C e^{\beta_2 z_2} = (B + z_2)\beta_2 + C \tag{13}$$

since the slope (with respect to β_2) at $\beta_2 = 0$ of the left-hand-side (resp., right-hand-side) of (13) is $z_2 C$ (resp., $B + z_2$). Using

$$d_2 = \frac{C e^{\beta_2 z_2}}{\beta_2} \tag{14}$$

(13) can be re-written as

$$d_2(1 - e^{-\beta_2 z_2}) = B + z_2 \tag{15}$$

Thus, our preceding discussion shows the existence of $\beta_2 > 0$ satisfying (15). The corresponding value of d_2 can now be obtained from (14). We thus have $\Pi_2^z = d_2(1-e^{-\beta_2 x_i})$ satisfying Property 3 (with respect to Π_1^z) and Properties 1, 2. The construction of the remaining functions $\Pi_i^z, i = 3, 4, ..., 3u$, is similar. ■

Appendix C: Proofs of Theorem 2, Lemma 2, and Theorem 3

Proof of Theorem 2: As with Problem P_1, 3-PARTITION can be reduced to the decision problem corresponding to P_2. The construction is exactly the same as in the proof of Theorem 1 with the following changes to ensure that the schedule of Figure 10 remains the unique optimum: (1) the reconfiguration time r satisfies $0 < r < \min_i\{\frac{B+z_i}{C} - z_i\}$. Note that $\frac{B+z_i}{C} - z_i > 0, i = 1, 2, ..., 3u$ (refer to Appendix B), (2) the planning horizon, $L = u(B+1) + 4ur$, (3) the time-window of Satellite $S_i^g, i = 1, 2, \ldots, u$, is $[a_i^g, b_i^g] = [i(B+4r+1) - r - 1, i(B+4r+1)]$ and the time-window of Satellite $S_i^z, i = 1, 2, \ldots, 3u$, is $[a_i^z, b_i^z] = [0, L]$, and (4) the planning horizon L' for Problem R_1 is $L' = uB + 3ur$. ■

Proof of Lemma 2: Without loss of generality, consider the situation illustrated in Figure 11: The time window for Satellite S_j starts at time q_k; satellites S_i and S_j are concurrently visible during the interval t_{k+1}. Let c_i and c_j be the associated linear coefficients of the utility functions for satellites S_i and S_j, respectively. Suppose in an optimum solution ρ, the switching of support from Satellite S_i to Satellite S_j takes place at $a, 0 < a < q_{k+1} - q_k$, units after Satellite S_j becomes visible at time q_k (see Figure 11).

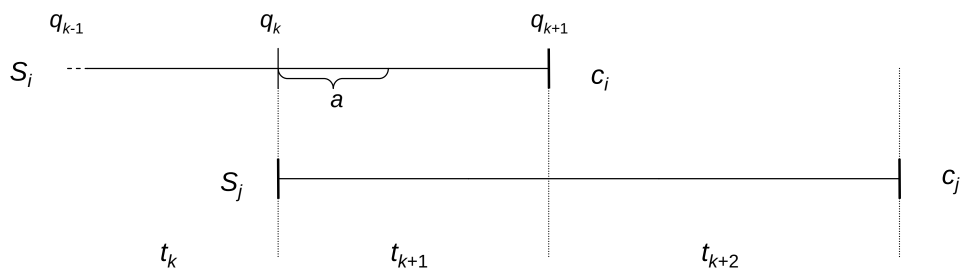

Figure 11: Satellite Switching Illustrating Lemma 2

Case 1: $c_i = c_j$. In this case, the total utility obtained by switching at time $q_k - r$ or at the end of interval t_k (i.e., at time q_k) or at time $q_{k+1} - r$ or at the end of interval t_{k+1} (i.e., at time q_{k+1})

is same as the total utility obtained by switching at time $q_k + a$. Hence, we obtain an alternate optimum solution ρ' by switching at time $q_k - r$, q_k, $q_{k+1} - r$ or q_{k+1} rather than at time $q_k + a$.

Case 2: $c_i < c_j$. If the switching took place at time $q_k - r$, the total utility would be $[(c_j - c_i)(a+r)] > 0$ units larger than the solution given by ρ. Thus, ρ cannot be an optimum solution.

Case 3: $c_i > c_j$. If the switching took place at the end of the interval t_{k+1} (i.e., at time q_{k+1}), the total utility would be larger than that of ρ by $[(c_i - c_j)(q_{k+1} - q_k - a)] > 0$ units. This, thus, contradicts the optimality of ρ. ■

Proof of Theorem 3: First, by Lemma 2, it is sufficient to optimize over the set of schedules represented in G. Suppose a longest path in G visits Satellite S_i during some interval t_k, then visits Satellite S_j during intervals t_{k+1} through t_{l-1}, and then revisits S_i again during interval t_l. Since the support switched from S_i to S_j in interval t_{k+1}, the utility (per unit time) offered by Satellite S_j is strictly higher than that of Satellite S_i (recall our assumption that $c_i \neq c_j, i \neq j, 1 \leq i, j \leq m$). Also, since $[a_i, b_i] \nsubseteq [a_j, b_j], i \neq j, 1 \leq i, j \leq m$, Satellite S_j must be available either during interval t_k, or interval t_l, or both; replacing the support of S_i by S_j in such an interval provides a strictly higher profit, thereby contradicting the optimality of the path. The proof is similar for the more general case where the longest path visits an arbitrary subset of satellites before revisiting S_i. ■

Appendix D: Proofs of Lemma 3 and Theorem 5

Proof of Lemma 3: Recall that $M = \{S_1, S_2, ..., S_m\}$. Initially, the set Ω is empty. We now show that at least one new satellite is added to Ω in Step 3 during each iteration. Suppose none of the satellites in Γ is included in Ω in Step 3 of an iteration. For simplicity of exposition, we assume $\Gamma = \{S_k\}$. As before, let μ_k be the set of satellites serviced during the time-window $[a_k, b_k]$ of S_k, and let the optimum value of the LP in Step 1 be y. Note that $y = y_k$ as $S_k \in \Gamma$, where y_k is the log-slope of S_k at the corresponding support-time x_k. Since S_k is not included in Ω at Step 3, there exists $S_i \in \mu_k, i \neq k$, whose log-slope (at its corresponding support-time x_i) is $y_i < y$.

Consider the following perturbation of the current solution in the time-window $[a_k, b_k]$: for an infinitesimally small $\theta > 0$, we decrease (resp. increase) the support of S_i (resp. S_k) in

$[a_k, b_k]$ by θ. The resulting feasible solution has an objective function value smaller than y, thus contradicting the optimality of y. The proof for $|\Gamma| > 1$ is similar. Since $|M| = m$, the result follows. ∎

Proof of Theorem 5: As shown in Lemma 4, Algorithm OptP$_3$ provides a solution in polynomial-time. We now show that this solution is, in fact, optimum. Let the objective function values obtained at ℓ^{th} iteration of Step 1 be $y^{\ell *}$. From Lemma 5, we have $y^{1*} \geq y^{2*} \geq \ldots \geq y^{u*}$, $u \leq m$. Let the corresponding satellite support-times be $x_j^*, j = 1, 2, ..., m$. Without loss of generality, we assume that the log-slopes (y_j^*) corresponding to the support-times x_j^* are in non-increasing order: $y_1^* \geq y_2^* \geq \ldots \geq y_m^*$. Note that if support-time x_j^* of Satellite S_j is determined by Algorithm OptP$_3$ at the end of ℓ^{th} iteration of Steps 1 and 3, then $j \geq \ell$ and $y_j^* = y^{\ell *}$.

Consider an optimum vector, $x^o = (x_1^o, x_2^o, \ldots, x_m^o)$, of the support-times. We will show that $x^o = x^* \equiv (x_1^*, x_2^*, \ldots, x_m^*)$. It is important to note that this does *not* imply the uniqueness of the optimum solution: the support-time x_j^* is the aggregate support-time for Satellite S_j. Showing $x^o = x^*$ proves that the aggregate support-times for the satellites are the same in any optimum solution. There can, however, be multiple schedules that generate the same aggregate support-times for all the satellites.

To proceed with our proof, let y_j^o denote the log-slopes corresponding to the support-times $x_j^o, i = 1, 2, ..., m$. Let S_p be the first satellite for which $x_p^o \neq x_p^*$; thus $x_r^o = x_r^*$, $r = 1, 2, \ldots, p-1$. We consider two cases: $x_p^o < x_p^*$ and $x_p^o > x_p^*$.

- **Case 1:** $x_p^o < x_p^*$. Then, $y_p^o > y_p^*$. Let $\epsilon > 0$, and $x_p^o = x_p^* - \epsilon$. From Lemmas 6 and 7, we have $x_1^o + x_2^o + \ldots + x_m^o = x_1^* + x_2^* + \ldots + x_m^* = L$. Let $q \geq p+1$ be the smallest index such that $x_q^o = x_q^* + \delta$, $\delta > 0$. Clearly, $y_p^* \geq y_q^*$. Let $0 < \theta \leq \min\{\epsilon, \delta\}$. Since $x_p^o < x_p^*$ and $x_q^o > x_q^*$, we have $y_p^o > y_p^*$ and $y_q^* > y_q^o$. Thus, $y_p^o > y_p^* \geq y_q^* > y_q^o$. Keeping all other variables fixed, consider the solution obtained by increasing (resp. decreasing) x_p^o (resp. x_q^o) by θ. That is, consider the feasible solution vector

$$(x_1^o, \ldots x_{p-1}^o, x_p^o + \theta, x_{p+1}^o, \ldots, x_{q-1}^o, x_q^o - \theta, x_{q+1}^o, \ldots, x_m^o)$$

Since $y_p^o > y_q^o$, the increase in utility corresponding to an increase of θ in the support-time of S_p is larger than the decrease in utility corresponding to the decrease of θ in the support-time of S_q. This contradicts the optimality of x^o.

- **Case 2:** $x_p^o > x_p^*$. This implies that $y_p^o < y_p^*$. Let $\epsilon > 0$ and $x_p^o = x_p^* + \epsilon$. As before, let $q \geq p+1$ be the smallest index with $x_q^o = x_q^* - \delta$, $\delta > 0$. Let $0 < \theta \leq \min\{\epsilon, \delta\}$. We consider two subcases:

  - **Case 2a:** $y_p^* > y_q^*$. Assume that the support-times of satellites $S_1, S_2, \ldots, S_k, k \geq j$, are fixed until the end of iteration ℓ. We, therefore, have $x_r^o = x_r^*$, $r = 1, 2, \ldots, p-1$, $x_p^o > x_p^*$, $x_r^o \geq x_r^*$, $r = p+1, \ldots, k, \ldots, q-1$, and $x_q^o < x_q^*$. The solution to the LP, corresponding to the first k satellites, in iteration ℓ is $(x_1^*, x_2^*, \ldots, x_p^*, \ldots, x_k^*)$. However, since $y_p^o < y_p^*$, the vector $(x_1^o, x_2^o, \ldots, x_p^o, \ldots, x_k^o)$ is a better solution (in iteration ℓ). We, therefore, have a contradiction.

  - **Case 2b:** $y_p^* = y_q^*$. An argument similar to that in Case 1 shows that $y_p^o < y_p^* = y_q^* < y_q^o$. As in Case 1, consider the feasible solution vector

    $$(x_1^o, \ldots x_{p-1}^o, x_p^o - \theta, x_{p+1}^o, \ldots, x_{q-1}^o, x_q^o + \theta, x_{q+1}^o, \ldots, x_m^o)$$

    As $y_p^o < y_q^o$, the increase in utility corresponding to an increase of θ in the support-time of S_q is larger than the decrease in utility corresponding to the decrease of θ in the support-time of S_p. This contradicts the optimality of x^o.

Since the choice of p was arbitrary, we have that $x_j^o = x_j^*$, $j = 1, 2, \ldots m$. The result follows. ■

Appendix E: Proof of Theorem 7

Proof of Theorem 7: Consider an arbitrary instance of Even-Odd Partition (EOP) (Garey and Johnson, 1979).

Even-Odd Partition: Given a set $A = \{z_1, z_2, \ldots, z_{2u-1}, z_{2u}\}$ and $z_i \in Z^+$ for each $i = 1, 2, \ldots, 2u$, where $z_1 < z_2 < \ldots < z_{2u-1} < z_{2u}$ and $\sum_{z_i \in A} z_i = 2B$, does there exist a partition of A into subsets A_1 and A_2 such that $\sum_{z_k \in A_1} z_k = \sum_{z_k \in A_2} z_k = B$, and that each of A_1, A_2 contains exactly one of z_{2i-1}, z_{2i} for $i = 1, 2, \ldots, u$?

Given an instance of Even-Odd Partition (EOP), we can construct a specific instance of the decision problem for Problem P_4 as follows (see Figure 12).

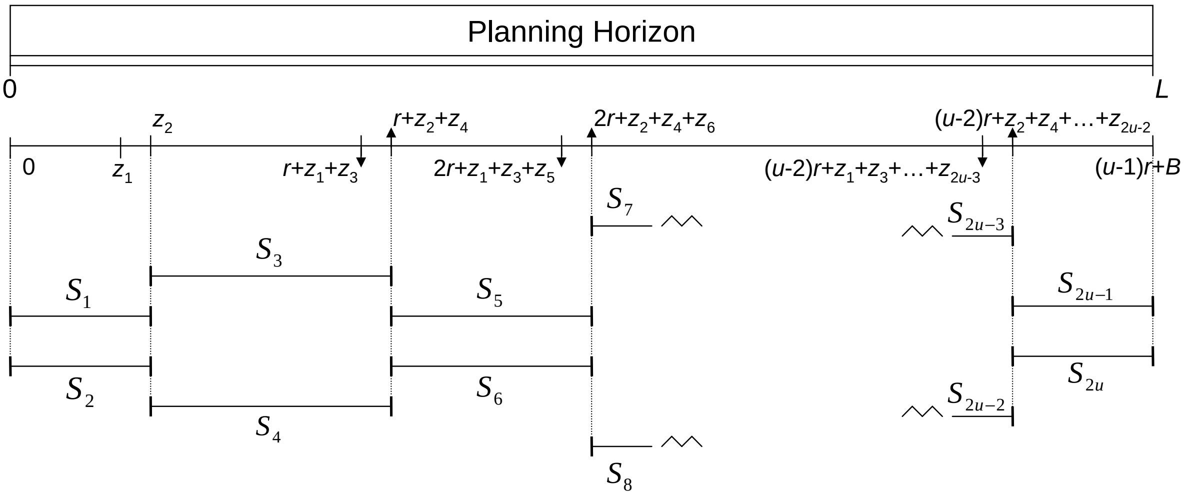

Figure 12: An Instance of the Decision Problem Q_4

- Consider a set of $2u$ satellites $M = \{S_i | 1 \leq i \leq 2u\}$.
- There is a positive reconfiguration time (r) of duration $2B$.
- The upper bound of the planning horizon, L, equals $(u-1)r + B$.
- The time-window, $[a_i, b_i]$, of S_i, $i = 1, 2, \ldots, 2u$, is defined as follows: $[a_1, b_1] = [a_2, b_2] = [0, z_2]$; $[a_{2j-1}, b_{2j-1}] = [a_{2j}, b_{2j}] = [(j-2)r + z_2 + \ldots + z_{2j-2}, (j-1)r + z_2 + \ldots + z_{2j}], j = 2, 3, \ldots, u-1$; $[a_{2u-1}, b_{2u-1}] = [a_{2u}, b_{2u}] = [(u-2)r + z_2 + \ldots + z_{2u-2}, (u-1)r + B]$.
- The properties of the utility functions $\Pi_i(x_i), i = 1, 2, ..., 2u$, are similar to those of the functions Π_i^z in the proof of Theorem 1 in Appendix A (see Figure 9), and are as defined below.

  1. $\Pi_i(x_i)$, $i = 1, 2, \ldots, 2u$, are differentiable and strictly concave.
  2. $\Pi_i(z_i) = K + z_i$, $i = 1, 2, \ldots, 2u$, where $K > 0$.
  3. $\Pi'_i(z_i) = \Pi'_j(z_j)$, $1 \leq i, j \leq 2u$, where $\Pi'^z_i(z_i)$ is the value of the derivative of the utility function of Satellite S_i at the support-time z_i. Without loss of generality, we assume that this common slope is 1. That is, $\Pi'_i(z_i) = 1, 1 \leq i \leq 2u$.

The proof of the existence of functions that obey the above properties is similar to that of Theorem 1 (see Appendices A and B).

For this specific instance of Problem P_4, consider the following decision problem (see Figure 12).

<u>**Decision Problem (Q_4):**</u> Does there exist a schedule σ of support-times such that the total utility over the planning horizon $[0, L]$ is at least $uK + B$?

The decision problem Q_4 is clearly in class NP. Also, it is easy to verify that the construction of Q_4 can be done in polynomial-time. We now show that there is a schedule σ of support-times such that $\Pi_\sigma \geq uK + B$ if and only if there exists a solution to Problem EOP.

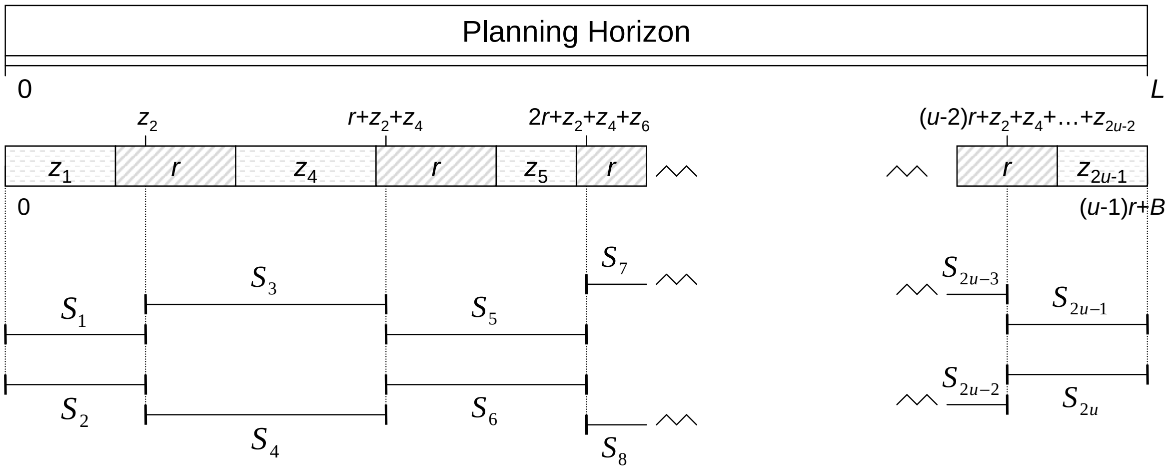

Figure 13: Proposed Schedule σ

<u>*If part:*</u> If there exists an EOP, we prove that there is a support schedule σ with a total utility $\Pi_\sigma = uK + B$. Assume $\sum_{z_k \in A_1} z_k = \sum_{z_k \in A_2} z_k = B$. Note that $M = \{S_1, S_2, \ldots, S_{2u}\}$. The time-windows of the satellites S_{2j-1} and S_{2j}, $j = 1, 2, \ldots, u$, are the same. There are u such disjoint intervals. The proposed schedule σ is illustrated in Figure 13. Exactly one satellite S_{2j-1} (or S_{2j}), $j = 1, 2, \ldots, u$, is scheduled for the duration of z_{2j-1} (or z_{2j}) at each disjoint interval t_j, $j = 1, 2, \ldots, u$, where $t_1 = [0, z_2]$, $t_j = [(j-2)r + \sum_{r=1}^{j-1} z_{2r}, (j-1)r + \sum_{r=1}^{j} z_{2r}]$, $j = 2, 3, \ldots, u-1$, and $t_u = [(u-2)r + \sum_{r=1}^{u-1} z_{2r}, (u-1)r + B]$. Since there are $(u-1)$ reconfigurations and $L = (u-1)r + B$, the total support-time cannot be more than B, which is indeed $\sum_{z_k \in A_1} z_k = B$. Recall also that the utility of the satellite S_{2j-1} (or S_{2j}) for the support-time z_{2j-1} (or z_{2j}) is $\Pi_{2j-1}(z_{2j-1}) = K + z_{2j-1}$ (or $\Pi_{2j}(z_{2j}) = K + z_{2j}$). Thus, the total utility (Π_σ) equals $uK + B$. Note that the first satellite can be reconfigured before time zero, which is ignored here.

Only if part: If there exists a support schedule σ_0 with $\Pi_{\sigma_0} \geq uK + B$, we show that there is an EOP through a series of claims. Since the value of r is large, there is no advantage to preempt and resume the support for any of the satellites in schedule σ_0.

CLAIM 1. There cannot be more than $(u-1)$ reconfigurations during the planning horizon $[0, L]$.

Proof: Since $ur = 2uB$, $ur > L = (u-1)r + B = 2uB - B$. The result follows.

CLAIM 2. At most one of the two satellites S_{2j-1} and S_{2j} is supported at each disjoint interval t_j, $j = 1, 2, \ldots, u$.

Proof: Note that the duration of each interval cannot be more than $r + B = 3B$. Since $2r = 4B > 3B$, at each interval j, $j = 1, 2, \ldots, u$, there cannot be two reconfigurations to serve two satellites. Thus, at most one of the two satellites (i.e., S_{2j-1} or S_{2j}) can be supported at each interval $j = 1, 2, \ldots, u$.

Due to the above claims, at most u satellites can be scheduled. More precisely, at each interval t_j, at most one satellite $\hat{S}_j$, where $\hat{S}_j \in \{S_{2j-1}, S_{2j}\}$, with support-time $\hat{z}_j \geq 0$ can be engaged. Moreover, due to the durations of the intervals, the following constraints must also be satisfied:

$$\begin{aligned}
\hat{z}_1 &\leq z_2 \\
\hat{z}_1 + \hat{z}_2 &\leq z_2 + z_4 \\
\hat{z}_1 + \hat{z}_2 + \ldots + \hat{z}_j &\leq z_2 + z_4 + \ldots + z_{2j}, j = 3, 4, \ldots, u-1. \\
\hat{z}_1 + \hat{z}_2 + \ldots + \hat{z}_{u-1} + \hat{z}_u &\leq B
\end{aligned}$$

Recall that the utility function is continuous and an increasing function of the support-time. Thus, in order to maximize the total utility, the sum of the support-times $\hat{z}_1 + \hat{z}_2 + \ldots + \hat{z}_{u-1} + \hat{z}_u$ must be equal to B, where each $\hat{z}_j$ is strictly positive. Now, we consider the following problem.

Problem R$_4$: The same set of satellites $M = \{S_1, S_2, \ldots, S_{2u}\}$ is visible as in Problem Q$_4$. The functional form of the nonlinear utility function is also same as before. Let $r = 0$ and each satellite be visible throughout the entire planning horizon $[0, L']$, with $L' = B$. Find an optimum non-preemptive solution that maximizes the utility subject to the constraint that exactly u satellites – one satellite $\bar{S}_j$ from each set, $\{S_{2j-1}, S_{2j}\}$, $j = 1, 2, \ldots, u$ – are supported.

CLAIM 3. If the the optimum solution value (i.e., the maximum utility) of Problem R$_4$ is at least

$uK+B$, then the support-time for $\bar{S}_j, j=1,2,\ldots,u$ is $\bar{z}_j \in \{z_{2j-1}, z_{2j}\}$, with $\bar{z}_1+\bar{z}_2+\ldots+\bar{z}_u = B$.

Proof: Consider an optimum vector of support-times $z^o = (z_1^o, \ldots, z_u^o) \neq \bar{z} = (\bar{z}_1, \ldots, \bar{z}_u)$. Recall that the common slope $\bar{s}$ of all the u satellites at the support-times corresponding to $\bar{z}$ is equal to one. Furthermore, it follows from Lemma 1 and Remark 1 that (i) the slope of all the u satellites at the support-times corresponding to z^o is equal, say s^o, and (ii) $z_1^o + \ldots + z_u^o = B$. Consider the following three cases:

(a) $s^o < \bar{s} = 1$. Then, $z_j^o > \bar{z}_j$, $j = 1,2,\ldots,u$. The total utility corresponding to support-times z^o, $\Pi(z^o)$ can be expressed as follows: $\Pi(z^o) = \Pi(\bar{z}) + \Delta_1$, where $\Delta_1 = \sum_{j=1}^{u}[\Pi_j(z_j^o) - \Pi_j(\bar{z}_j)]$ and $\Pi(\bar{z}) = uK + \sum_{j=1}^{u} \bar{z}_j$. Since $\bar{s} = 1$, we have $\Pi_j(z_j^o) - \Pi_j(\bar{z}_j) < z_j^o - \bar{z}_j$, $j = 1,2,\ldots,u$. Thus, we have

$$\Pi(z^o) = \Pi(\bar{z}) + \Delta_1 < uK + \sum_{j=1}^{u} \bar{z}_j + (\sum_{j=1}^{u} z_j^o - \sum_{j=1}^{u} \bar{z}_j) = uK + B$$

This contradicts the assumption that z^o is an optimum solution with total utility at least $uK + B$.

(b) $s^o > \bar{s} = 1$. In this case, $z_j^o < \bar{z}_j$, $j = 1,2,\ldots,u$. The total utility corresponding to support-times z^o, $\Pi(z^o)$ can be expressed as follows: $\Pi(z^o) = \Pi(\bar{z}) - \Delta_2$, where $\Delta_2 = \sum_{j=1}^{u}[\Pi_j(\bar{z}_j) - \Pi_j(z_j^o)]$ and $\Pi(\bar{z}) = uK + \sum_{j=1}^{u} \bar{z}_j$. Since $\bar{s} = 1$, we have $\Pi_j(\bar{z}_j) - \Pi_j(z_j^o) > \bar{z}_j - z_j^o$, $j = 1,2,\ldots,u$. Thus,

$$\Pi(z^o) = \Pi(\bar{z}) - \Delta_2 < uK + \sum_{j=1}^{u} \bar{z}_j - (\sum_{j=1}^{u} \bar{z}_j - \sum_{j=1}^{u} z_j^o) = uK + B$$

This, again, contradicts the assumption that the total utility corresponding to z^o is at least $uK + B$.

(c) $s^o = \bar{s} = 1$. The contradictions obtained for the previous two cases imply that this case is the only possibility. It follows that $z_j^o = \bar{z}_j$, $j = 1,2,\ldots,u$. Then, the utility equals $\sum_{j=1}^{u} K + \bar{z}_j = uK + B$. Thus, $\sum_{j=1}^{u} \bar{z}_j = B$.

<u>Claim 4</u>. If $\Pi_{\sigma_0} \geq uK+B$, then $\hat{z}_j$, $j = 1,2,\ldots,u$, corresponds to an EOP, i.e., $\hat{z}_j \in \{z_{2j-1}, z_{2j}\}$, $j = 1,2,\ldots,u$.

Proof: Follows immediately from Claims 1-3. ■