Appendix

PROOF to PROPOSITION 3.1

Let \( \tilde{m}(t) = -\epsilon \tilde{m}(t) \), \( \tilde{m}(0) = m^0 \geq 0 \). Then \( \tilde{m}(t) = m^0 e^{-\epsilon t} \geq 0 \), for \( 0 \leq t \leq T \). Since \( ah(P_2, m, Q_2) g(0, Q_1) + h(P_2, m, Q_2) \geq 0 \), it is obvious that \( m(t) \geq \tilde{m}(t) \geq 0 \), where \( m(t) \) is a solution of (6) for arbitrary \( P_2(t) \) and \( Q_1(t) \). Since \( m(t) \geq 0 \), we can similarly conclude that \( Q_1(t) \geq 0 \) for \( 0 \leq t \leq T \). Moreover, it is easy to see that \( m^0 > 0 \) and \( Q_1^0 > 0 \) imply \( m(t) > 0 \) and \( Q_1(t) > 0 \) for \( 0 \leq t \leq T \). 

PROOF to PROPOSITION 3.2

(a) From (4), we see that \( V_\alpha^O(0, Q_1^0, m^0) = J(P_2^0(t | \alpha)) \) or simply \( V_\alpha^O = J(P_2^0(t | \alpha)) \), where \( P_2^0(t | \alpha) \) is the optimal price trajectory given \( \alpha \). Let \( Q_1^0(t | \alpha) \) denote the software quality trajectory given a price trajectory \( P_2(t) \) and \( \alpha \). Similarly, let \( m^0(t | \alpha) \) denote the user network size trajectory given a price trajectory \( P_2(t) \) and \( \alpha \).

From (5), we have \( Q_1(t) = Q_1^0 e^{-\epsilon t} + \alpha e^{-\epsilon t} \int_0^t e^{\epsilon \tau} m(\tau) d\tau \), which increases with \( \alpha \) for every fixed trajectory \( m(t) \). Next we see from (6) that for a given price trajectory \( P_2(t) \geq 0 \), \( \partial m(t) = ah(P_2(t), m(t), Q_2(t)) \geq 0 \), which means that \( m(t) \) increases with \( Q_1(t) \). This implies that both \( Q_1(t) \) and \( m(t) \) increase as \( \alpha \) increases for a given price trajectory \( P_2(t) \geq 0 \).

Let \( 0 < \alpha_1 < \alpha_2 \). Then \( Q_1^0(t | \alpha_1) \leq Q_1^0(t | \alpha_2) \) and \( m^0(t | \alpha_1) \leq m^0(t | \alpha_2) \). By the assumption on the functions \( h \) and \( g \), it is apparent that \( J(P_2(t | \alpha_1)) \leq J(P_2(t | \alpha_2)) \). By definition, \( J(P_2(t | \alpha_1) \leq J(P_2(t | \alpha_2)) \). Therefore, \( V_\alpha^O \leq V_\alpha^O \). This completes the proof.

(b) The proof is similar to part (a).

PROOF to PROPOSITION 3.3

From Proposition 3.2 part (b), we know that \( \frac{\partial V_\alpha^O(0, Q_1(0))}{\partial Q_1(0)} \geq 0 \). Therefore, \( \lambda(0) = \frac{\partial V_\alpha^O(0, Q_1(t))}{\partial Q_1(t)} \geq 0 \). The same argument extends to \( \lambda(t) = \frac{\partial V_\alpha^O(0, Q_1(t))}{\partial Q_1(t)} \geq 0 \).

PROOF to PROPOSITION 3.4

This proof requires Lemma 11.

**Lemma 1** In the open source model, \( P_2 + \mu(0, Q_1(1) + 1) \geq 0 \), for \( 0 \leq t \leq T \).

**Proof to Lemma 1**

According to (11) and (12), we know that there are two cases:

**Case 1:**

\[ \left\{ h(P_2, m, Q_2) + [P_2 + \mu(0, Q_1(1) + 1)] \frac{\partial h}{\partial P_2} \right\}_{P_2 = 0} \leq 0 \]

and \( \eta_2 \geq 0 \).

**Case 2:**

\[ \left\{ h(P_2, m, Q_2) + [P_2 + \mu(0, Q_1(1) + 1)] \frac{\partial h}{\partial P_2} \right\}_{P_2 > 0} = 0 \]

and \( \eta_2 = 0 \).

In case 1, \( [P_2 + \mu(0, Q_1(1) + 1)]_{P_2 = 0} \geq -h(P_2, m, Q_2) \geq 0 \).

In case 2, \( [P_2 + \mu(0, Q_1(1) + 1)]_{P_2 > 0} \geq -h(P_2, m, Q_2) \geq 0 \). The result follows.

By contradiction. Suppose at an arbitrarily chosen time \( \tau \in [0, T] \), \( \mu(\tau) < 0 \). By Proposition 3.3 and Lemma 11

\[ \dot{\mu} = (\rho + \epsilon) \mu - \mu \lambda - [P_2 + \mu(0, Q_1(1) + 1)] \frac{\partial h}{\partial \mu} < 0 \]

Therefore, \( \mu(\tau) < 0 \) for \( \tau \leq t \leq T \). This contradicts \( \mu(T) \geq 0 \). So \( \mu(\tau) \geq 0 \). Since \( \tau \) is arbitrary, we can conclude that \( \mu(T) \geq 0 \) for \( 0 \leq t \leq T \).

**Proof to Proposition 3.5**

If \( P_2^* > 0 \), then \( h + P_2 \frac{\partial h}{\partial P_2} + \mu(0, Q_1(1)) \frac{\partial h}{\partial P_2} = 0 \) (from (11)). By Proposition 3.4 and the assumptions that \( \mu \geq 0 \) and \( \frac{\partial h}{\partial P_2} \leq 0 \), we have \( \mu(0, Q_1(1)) \frac{\partial h}{\partial P_2} \leq 0 \).

Therefore, \( h + P_2 \frac{\partial h}{\partial P_2} \geq 0 \). By definition, \( \frac{\partial F_\alpha}{\partial P_2} |_{P_2^*} = h + P_2 \frac{\partial h}{\partial P_2} |_{P_2^*} = 0 \). Then

\[ \dot{P}_2 = h + P_2 \frac{\partial h}{\partial P_2} |_{P_2^*} \geq 0 \].

By the concavity of \( F_\alpha \), we conclude \( P_2^*(t) \leq \dot{P}_2(m^*(t), Q_2(t)) \) for \( 0 \leq t \leq T \). Moreover, if the salvage value is zero at time \( T \), then \( \mu(T) = 0 \). From the previous argument, it is easy to show \( P_2(T) = \dot{P}_2(m^*(T), Q_2(T)) \).
PROOF to PROPOSITION 3.6 The proof is similar to that of Proposition 3.1.

PROOF to PROPOSITION 3.7

The proofs for part (a) and (b) are similar to that of Proposition 3.2. (c) Using the Envelope Theorem (e.g., Varian, 1978, Page 268), we have \( \frac{dV_C}{dw} = \frac{dL}{dw} = -\int_0^T N^2 dt \). Therefore, the optimal closed source profit decreases with \( w \).

PROOF to PROPOSITION 3.8 The proof is similar to that of Proposition 3.5.

PROOF to PROPOSITION 3.9

The proof requires Lemma 2 and Proposition 3.8.

**LEMMA 2.** In the closed source model, \( P_1 + \mu \geq 0 \) and \( P_2 + \mu + (P_1 + \mu)g(P_1, Q_1) \geq 0 \), for \( 0 \leq t \leq T \).

PROOF to LEMMA 2 The proof is similar to that of Lemma 1.

By contradiction. Suppose at an arbitrarily chosen time \( \tau \in [0, T] \), \( \mu(\tau) < 0 \). By Proposition 3.8 and Lemma 2,

\[ \mu = (\rho + \epsilon)\mu - [P_2 + \mu + (P_1 + \mu)g(P_1, Q_1)] \frac{\partial h}{\partial \mu} < 0. \]

Therefore, \( \mu(\tau) < 0 \) for \( \tau < t \leq T \). This contradicts \( \mu(T) \geq 0 \). So \( \mu(\tau) \geq 0 \). Since \( \tau \) is arbitrary, we can conclude that \( \mu(T) \geq 0 \) for \( 0 \leq t \leq T \).

PROOF to PROPOSITION 3.10

If \( P_1^* > 0 \), then

\[
\left. \begin{array}{l}
\frac{dP_1}{dP_1} h(P_1, m, Q_2) + a \mu \frac{\partial g}{\partial P_1} \\
\frac{\partial g}{\partial P_1} h(P_2, q, Q_2) \\
\end{array} \right|_{P_1^*} = 0 \text{ (from (20))}.
\]

We can also say that \( g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1} + a \mu \frac{\partial g}{\partial P_1} \bigg|_{P_1} = 0 \). By Proposition 3.9 and the assumption that \( \frac{\partial g}{\partial P_1} \leq 0 \), we have

\[ a \mu \frac{\partial g}{\partial P_1} \bigg|_{P_1} < 0. \]

Therefore, \( g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1} \bigg|_{P_1} \geq 0 \).

By definition,

\[ \frac{\partial F_C}{\partial \tau} \bigg|_{\hat{P}_1, \hat{P}_2} = \left( g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1} \right) h(P_2, m, Q_2) \bigg|_{\hat{P}_1, \hat{P}_2} = 0. \]

We can also say that \( g(P_1, Q_1) + P_1 \frac{\partial g}{\partial P_1} \bigg|_{\hat{P}_1} = 0 \). Then \( \hat{P}_1 = -g(P_1, Q_1) \bigg|_{\hat{P}_1} \geq 0 \).

By the concavity of \( F_C \), we conclude \( P_1^*(t) \leq \hat{P}_1(t) \) for \( 0 \leq t \leq T \).

PROOF to COROLLARY 3.1

(i) Exponential demand function. From Proposition 4.1, \( P_1^*(t) \leq m^*(t)Q_2(t) \). From (6), \( \hat{m} = a \exp(-m + \epsilon) + 1 ) \exp(-m + \epsilon) \) - \( \hat{m}m(0) = m^0 \). Let \( \hat{m} = (a + 1) - \epsilon, \hat{m}m(0) = m^0. \) Then \( \hat{m}(t) = m^0e^{-\epsilon t} + (a + 1)(1 - e^{-\epsilon t})/\epsilon, 0 \leq t \leq T \). Clearly, \( \hat{m} > \hat{m} \).

(ii) Linear-price demand function. Proof is similar to (i).

PROOF to COROLLARY 3.2

(i) Exponential demand function. From Proposition 4.1, \( P_1^*(t) \leq Q_1^*(t) \). From (14), \( Q_1 = kN - \delta Q_1, Q(0) = Q_0^* \). Let \( \hat{Q}_1 = kN(0) - \delta \hat{Q}_1, \hat{Q}_1(0) = Q_0^* \), where \( N(0) \) is the number of in-house programmers at time \( 0 \). Then \( \hat{Q}_1(t) = Q_0^*e^{-\delta t} + kN(0)(1 - e^{-\delta t})/\delta, 0 \leq t \leq T \). Clearly, \( \hat{Q}_1 > Q_1 \) since \( N \) is decreasing over time.

The result follows.

From Proposition 4.1, \( P_1^*(t) \leq m^*(t)Q_2(t) - \hat{P}_1(t)g(\hat{P}_1(t), Q_1(t)) \leq m^*(t)Q_2(t) \). From (13), \( m = m^0...\)
\[ (a \exp(-\frac{P_1 + c}{Q_1}) + 1) \exp(-\frac{P_2}{mQ_2}) - \epsilon m, m(0) = m^0. \]

Let \( \dot{\hat{m}} = (a + 1) - \epsilon m, \hat{m}(0) = m^0. \) Then \( \hat{m}(t) = m^0 e^{-\epsilon t} + (a + 1)(1 - e^{-\epsilon t})/\epsilon, \) \( 0 \leq t \leq T. \) Clearly, \( \dot{\hat{m}} > \dot{\hat{m}}. \) The result follows.

(ii) Linear-price demand function. Proof is similar to (i). \( \square \)