Flexible Backup Supply and the Management of Lead-Time Uncertainty

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1 Appendix

Proof of Proposition 1. 1) With a little algebra, we can get, if $1 > r > 0$ holds, then

$$R(\beta| r) = \frac{1}{2} \frac{(\pi + h)^2 - h^2}{\pi + h} \left( \beta T - \frac{(\pi r T - (c_f - c_d))}{(\pi + h)^2 - h^2} (\pi + h) \right)^2 + \frac{1}{2} \pi T^2$$

$$+ \frac{1}{2} h (1 - r)^2 T^2 - \frac{1}{2} \frac{(\pi r T - (c_f - c_d))^2}{(\pi + h)^2 - h^2} (\pi + h)$$

and if $r \geq 1$ holds, then

$$R(\beta| r) = \frac{1}{2} \frac{(\pi + h)^2 - h^2}{\pi + h} \left( \beta T - \frac{(\pi r T - (c_f - c_d))}{(\pi + h)^2 - h^2} (\pi + h) \right)^2 + \pi r T^2$$

$$- \frac{1}{2} \pi T^2 - \frac{1}{2} \frac{(\pi r T - (c_f - c_d))^2}{(\pi + h)^2 - h^2} (\pi + h)$$

Therefore $R(\beta| r)$ is minimized when $\beta T - \frac{(\pi r T - (c_f - c_d))}{(\pi + h)^2 - h^2} (\pi + h) = 0$. This leads to our part 1) conclusion in view of the boundary conditions for $\beta$.

2) $\beta^* > 0$ hold if and only if $\pi r T - (c_f - c_d) > 0$; and $\beta^* < 1$ hold if and only if $\pi r T - (c_f - c_d) < 0$. This leads to our part 2) conclusion.

3) Part 3) conclusion is true because $(c_f - c_d) > 0$ and $\frac{\pi + h}{\pi + 2h} < 1$.

Proof of Algorithm 1. We first examine the situations where $l_f \leq T$ holds. We will analyze the cases defined in (8). For the case $\beta < \frac{l_f}{T}$, $r \leq 1$, since it is obvious that the optimal $l_f$ is $l_f$, we focus on the decision for $\beta$. It can be seen that $R\left(\beta, l_f^*| r\right)$ is linear in $\beta$ with the first order
derivative
\[
\frac{\partial R(\beta, l_f^r | r)}{\partial \beta} = T \left( -\pi \left( rT - l_f \right) + (c_f - c_d) \right)
\]
Thus, when \( rT \leq \frac{c_f - c_d}{\pi} + l_f \), the optimal \( \beta \) is 0; when \( rT > \frac{c_f - c_d}{\pi} + l_f \), the optimal \( \beta \) is \( \frac{l_f}{T} \).

For the case \( \beta \geq \frac{l_f}{T}, r \leq 1 \), it can be seen that \( R(\beta, l_f | r) \) is convex in \( l_f \) with the first order derivative
\[
\frac{\partial R(\beta, l_f | r)}{\partial l_f} = (\pi + h) l_f - h\beta T
\]
Therefore the decision rule on \( l_f \) for given \( \beta \) is: to choose \( l_f = \frac{h}{\pi + h} \beta T \) if \( \frac{h}{\pi + h} \beta T > l_f \), and to choose \( \frac{l_f}{T} \) otherwise. The value of \( R(\beta, l_f | r) \) at the optimal \( l_f \), denoted by \( R(\beta, l_f^* | r) \), is accordingly given below
\[
R(\beta, l_f^* | r) = (c_f - c_d) \beta T + \frac{1}{2} \pi (rT - \beta T)^2 + \frac{1}{2} h (T - rT)^2
\]
\[
+ \begin{cases} 
\frac{1}{2} \pi \left( l_f \right)^2 + \frac{1}{2} h \left( \beta T - l_f \right)^2 & \text{if } \beta T > l_f, \frac{h}{\pi + h} \beta T < l_f, r \leq 1 \\
\frac{1}{2} \pi \left( \beta T \right)^2 & \text{if } \beta T > l_f, \frac{h}{\pi + h} \beta T > l_f, r \leq 1
\end{cases}
\]
The first order derivative for \( R(\beta, l_f^* | r) \) with respect to \( \beta \) can be obtained as follows
\[
\frac{dR(\beta, l_f^* | r)}{d\beta} = \begin{cases} 
\left( (c_f - c_d) + (\pi + h) \beta T - hl_f - \pi rT \right) T & \beta T > l_f, \frac{h}{\pi + h} \beta T < l_f, r \leq 1 \\
\left( (c_f - c_d) + \frac{h}{\pi + h} \beta T + \pi \beta T - \pi rT \right) T & \beta T > l_f, \frac{h}{\pi + h} \beta T > l_f, r \leq 1
\end{cases}
\]
Based on the expression above, it can be seen that with a little algebra, \( R(\beta, l_f^* | r) \) is convex in \( \beta \) over \([0, r]\) for given \( r \). Therefore the optimal \( \beta \) can be determined from the first order condition given above. Particularly, we have: a) if \( rT < \frac{c_f - c_d}{\pi} + l_f \), then the optimal \( \beta \) is 0. This is because \( \frac{dR(\beta, l_f^* | r)}{d\beta} > 0 \) for \( \beta \in [0, r] \); b) if \( rT \) is greater than \( \frac{c_f - c_d}{\pi} + l_f \) and less than \( \frac{c_f - c_d}{\pi} + \frac{(\pi + h)^2 - h^2}{\pi h} l_f \), then the optimal \( \beta T \) is \( \frac{\pi}{\pi + h} \left( rT \right) + \frac{h}{\pi + h} l_f - \frac{c_f - c_d}{\pi} \), which is less than \( rT \). This is because \( \frac{dR(\beta, l_f^* | r)}{d\beta} \) is negative at \( \beta = \left( \frac{c_f - c_d}{\pi} + l_f \right) / T \) and is positive at \( \beta = \left( \frac{c_f - c_d}{\pi} + \frac{(\pi + h)^2 - h^2}{\pi h} l_f \right) / T \); c) if \( rT \) is greater than \( \frac{c_f - c_d}{\pi} + \frac{(\pi + h)^2 - h^2}{\pi h} l_f \), then the optimal \( \beta T \) is \( \frac{\pi + h}{(\pi + h)^2 - h^2} \left( \pi rT - (c_f - c_d) \right) \), which is less than \( rT \), since \( \frac{dR(\beta, l_f^* | r)}{d\beta} \) is negative at \( \beta = \left( \frac{c_f - c_d}{\pi} + \frac{(\pi + h)^2 - h^2}{\pi h} l_f \right) / T \).

Similar spirit above can be applied to analyze the cases for \( r > 1 \). For the case \( \beta < \frac{l_f}{T}, r > 1 \), it is obvious that the optimal \( l_f \) is \( l_f \). Regarding the decision for \( \beta \), we can get: if \( rT < \frac{c_f - c_d}{\pi} + l_f \) holds, then the optimal \( \beta \) is 0; if \( rT > \frac{c_f - c_d}{\pi} + l_f \) holds, then the optimal \( \beta \) is \( \frac{l_f}{T} \).

For the case \( \beta > \frac{l_f}{T}, r > 1 \), the optimal \( l_f \) is \( l_f \) if \( \frac{h}{\pi + h} \beta T < l_f \) holds, and is \( \frac{h}{\pi + h} \beta T \) otherwise.
The value of $R(\beta, l_f | r)$ at the optimal $l_f$, denoted by $R(\beta, l_f^* | r)$, is

$$
R(\beta, l_f^* | r) = (c_f - c_d) \beta T + \\
\begin{cases}
\frac{1}{2} \pi (l_f^*)^2 + \pi (l_f - \beta T) (rT - l_f^*) + \frac{1}{2} \pi (T - l_f^*)^2 + \pi (rT - T) (T - l_f^*) & \text{if } \beta < \frac{l_f^*}{T}, r > 1 \\
\frac{1}{2} \pi (T - \beta T)^2 + \pi (T - \beta T) (rT - T) + \frac{1}{2} \pi (l_f^*)^2 + \frac{1}{2} h (\beta T - l_f^*)^2 & \text{if } \beta T > l_f^*, \frac{h}{\pi + h} \beta T < l_f^*, r > 1 \\
\frac{1}{2} \pi (T - \beta T)^2 + \pi (T - \beta T) (rT - T) + \frac{1}{2} \pi (\frac{\pi h}{\pi + h} (\beta T))^2 & \text{if } \beta T > l_f^*, \frac{h}{\pi + h} \beta T \geq l_f^*, r > 1
\end{cases}
$$

Based on the expression above, we can get the decision of the optimal $\beta T$. Particularly, we have:

- If $rT < \frac{c_f - c_d}{\pi} + l_f^*$, then the optimal $\beta T$ is $0$.
- If $rT$ is between $\frac{c_f - c_d}{\pi} + l_f^*$ and $\frac{c_f - c_d}{\pi} + \frac{h}{\pi + h} l_f^*$, then the optimal $\beta T$ is $\frac{\pi + h}{\pi + h} (\pi rT - (c_f - c_d))$. All of the optimal $\beta T$ have to be bounded above by $T$.
- Particularly, in case $\frac{\pi + h}{(\pi + h)^2 - h^2} (\pi rT - (c_f - c_d)) > T$ and $rT > \frac{c_f - c_d}{\pi} + \frac{h}{\pi + h} l_f^*$, then the optimal $\beta T$ is $T$ with a cost of $(c_f - c_d) T + \frac{1}{2} \pi (l_f^*)^2$ for $R(\beta^*, l_f^* | r)$ if $\frac{h}{\pi + h} T \geq l_f^*$.
- And the optimal $\beta T$ is $T$ with a cost of $(c_f - c_d) T + \frac{1}{2} \pi (l_f^*)^2$ for $R(\beta^*, l_f^* | r)$ if $\frac{h}{\pi + h} T < l_f^*$. In case that $\frac{\pi + h}{\pi + h} (\pi rT) + \frac{h}{\pi + h} l_f^*$ is $T$ and $rT$ is between $\frac{c_f - c_d}{\pi} + l_f$ and $\frac{c_f - c_d}{\pi} + \frac{h}{\pi + h} l_f$, then the optimal $\beta T$ is $T$ with a cost of $(c_f - c_d) T + \frac{1}{2} \pi (l_f^*)^2$ for $R(\beta^*, l_f^* | r)$.

We now examine the situations where $l_f > T$ holds. It is obvious that the optimal $l_f$ is $l_f^*$.

Recall that

$$
R(\beta, l_f | r) = (c_f - c_d) \beta T + \frac{1}{2} \pi (\beta T)^2 + \pi (\beta T) (l_f - \beta T) + \frac{1}{2} \pi (T - \beta T)^2 + \pi (T - \beta T) (rT - T)
$$

It can be seen that with a little algebra, if $rT \leq \frac{c_f - c_d}{\pi} + l_f$, then the optimal $\beta T$ is zero with a cost of $\frac{1}{2} \pi T^2 + \pi T (rT - T)$ for $R(\beta^*, l_f^* | r)$; if $rT > \frac{c_f - c_d}{\pi} + l_f$, then the optimal $\beta T$ is $T$ with a cost of $(c_f - c_d) T + \frac{1}{2} \pi T^2$ for $R(\beta^*, l_f^* | r)$.

Putting all the above together yields the proof for Algorithm 1. ■

**Proof of Proposition 2.** With a little algebra, we can decompose $\nabla_{II} \left( Q_1, Q_2 | \xi_0, l, \bar{T} \right)$ as follows

$$
\nabla_{II} \left( Q_1, Q_2 | \xi_0, l, \bar{T} \right) = \nabla_{II}^1 \left( Q_1 | \xi_0, l, \bar{T} \right) + \nabla_{II}^2 \left( Q_2 | \xi_0, l, \bar{T} \right) + (c_f - c_d) \left( T - \bar{T} \right)
$$

(13)

where

$$
\nabla_{II}^1 \left( Q_1 | \xi_0, l, \bar{T} \right) = (c_f - c_d) Q_1 + \frac{1}{2} \pi \frac{h}{\pi + h} Q_1^2 + \frac{1}{2} \pi (\xi_0 - Q_1)^2
$$

(14)
\[ \nabla^2_{\Omega} \left( Q_2 | \xi_0, l, \tilde{T} \right) = - (c_f - c_d) Q_2 + \frac{\pi h}{\pi + h} \left( T - \tilde{T} - Q_2 \right)^2 \] 

\[ + \begin{cases} \frac{1}{2} h (T + Q_2 - \xi_0)^2 & \text{if } Q_1 < \xi_0 \leq \tilde{T} + Q_2 \\ - \frac{1}{2} \pi (T + Q_2 - \xi_0)^2 & \text{if } Q_1 \leq \tilde{T} + Q_2 < \xi_0 \end{cases} \] 

The first order derivatives of \( \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right) \) with respect to \( Q_1 \) and \( Q_2 \) are, respectively,

\[ \frac{\partial \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right)}{\partial Q_1} = (c_f - c_d) + \frac{\pi h}{\pi + h} Q_1 + \pi (Q_1 - \xi_0) \] 

\[ \frac{\partial \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right)}{\partial Q_2} = - (c_f - c_d) + \frac{\pi h}{\pi + h} \left( \tilde{T} + Q_2 - T \right) \]

Based on the expressions above (13), (14), (15), (16) and (17), we see that the following properties hold: 1) \( \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right) \) is separable in \( Q_1 \) and \( Q_2 \); and, \( \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right) \) is convex in \( Q_1 \); 2) \( \nabla_{\Omega} \left( Q_1, Q_2 | \xi_0, l, \tilde{T} \right) \) is concave in \( Q_2 \) for \( T + Q_2 < \xi_0 \) and is convex in \( Q_2 \) for \( Q_1 < \xi_0 \leq \tilde{T} + Q_2 \). Furthermore, by the expressions for \( Q_1 (\xi_0) \) and \( Q_2 (\xi_0) \) and the expressions above, it can be seen that \( Q^{UC} \approx (Q_1 (\xi_0), Q_2 (\xi_0)) \) is the unique local minimizer of (10) without constraints.

If \( Q_1 = Q_2 \), then the first-order derivative of \( \nabla_{\Omega} \left( Q_2, Q_2 | \xi_0, l, \tilde{T} \right) \) is

\[ \frac{\partial \nabla_{\Omega} \left( Q_2, Q_2 | \xi_0, l, \tilde{T} \right)}{\partial Q_2} = \frac{\pi h}{\pi + h} Q_2 + \frac{\pi h}{\pi + h} \left( \tilde{T} + Q_2 - T \right) + \pi (Q_2 - \xi_0) \]

\[ + \begin{cases} h \left( \tilde{T} + Q_2 - \xi_0 \right) & \text{if } Q_2 < \xi_0 \leq \tilde{T} + Q_2 \\ - \pi \left( \tilde{T} + Q_2 - \xi_0 \right) & \text{if } Q_2 \leq \tilde{T} + Q_2 < \xi_0 \end{cases} \] 

The expression above implies that \( \nabla_{\Omega} \left( Q_2, Q_2 | \xi_0, l, \tilde{T} \right) \) is piecewise convex in \( Q_2 \). Based on (18), we can get the expression for the minimizer of \( \nabla_{\Omega} \left( Q_2, Q_2 | \xi_0, l, \tilde{T} \right) \). This turns out that \( Q^{OA} \approx (Q_2^{OA}, Q_2^{OA}) \) is the minimizer of \( \Gamma_{\Omega} \left( Q_2, Q_2 | \xi_0, l, \tilde{T} \right) \).

If \( Q_1 = \tilde{T} + Q_2 \), then \( \nabla_{\Omega} \left( \tilde{T} + Q_2, Q_2 | \xi_0, l, \tilde{T} \right) \) has an expression

\[ (c_f - c_d) \tilde{T} + (c_f - c_n) \left( T - \tilde{T} \right) + \frac{1}{2} \frac{\pi h}{\pi + h} \left( \tilde{T} + Q_2 \right)^2 + \frac{1}{2} \frac{\pi h}{\pi + h} \left( T - \tilde{T} - Q_2 \right)^2 \]
which is convex in \( Q_2 \). It can be easily verified that \( Q^{BC} = (\bar{T} + Q_2^{BC}, Q_2^{BC}) \) is the minimizer of \( \nabla_{\Pi} \left( \bar{T} + Q_2, Q_2 \mid \xi_0, l, \bar{T} \right) \). Similarly it can be shown that if \( Q_2 = 0 \), then \( \nabla_{\Pi} \left( Q_1, 0 \mid \xi_0, l, \bar{T} \right) \) is minimized at \( Q^{CO} = (Q_1^{CO}, 0) \) satisfying

\[
Q_1^{CO} = \begin{cases} 
0 & \text{if } \pi \xi_0 \leq (c_f - c_d) \\
\frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} & \text{if } 0 \leq \frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} < \bar{T} \\
\bar{T} & \text{if } \frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} \geq \bar{T}
\end{cases}
\]

and that if \( Q_2 = T - \bar{T} \), then \( \nabla_{\Pi} \left( Q_1, T - \bar{T} \mid \xi_0, l, \bar{T} \right) \) is minimized at \( Q^{AB} = (Q_1^{AB}, T - \bar{T}) \) satisfying

\[
Q_1^{AB} = \begin{cases} 
T - \bar{T} & \text{if } \frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} \leq T - \bar{T} \\
\frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} & \text{if } T - \bar{T} \leq \frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} < T \\
T & \text{if } \frac{\pi \xi_0 - (c_f - c_d)}{\pi + \pi} \geq T
\end{cases}
\]

Now, we are ready to show Proposition 2 is valid.

1). Since \( \xi_0 \geq T \), \( Q_2 + \bar{T} \leq \xi_0 \) holds for any \( Q_2 \leq T - \bar{T} \). Thus \( \nabla_{\Pi} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) is concave in \( Q_2 \). For any \( Q_1 \), \( \nabla_{II} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) could be minimized only at the boundary points of the feasible set \( OABC \). The minimum of \( \nabla_{II} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) could be achieved only at the four sides of the feasible set \( OABC \) illustrated in Figure ???. Since the minimum of \( \nabla_{III} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) on the four sides could only be achieved at one of the four points \( Q^{OA}, Q^{CO}, Q^{BC} \) and \( Q^{AB} \), respectively, part 1) follows.

2). Since \( \xi_0 < T \), there may exist \( Q_2 \) such that \( Q_2 + \bar{T} > \xi_0 \) holds. Thus \( \nabla_{II}^2 \left( Q_2 \mid \xi_0, l, \bar{T} \right) \) is concave-convex in \( Q_2 \). For any \( Q_1 \), \( \nabla_{II} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) could be minimized only at the boundary points of the feasible set \( OABC \) or \( Q_2 (\xi_0) \). If \( Q^{UC} = (Q_1 (\xi_0), Q_2 (\xi_0)) \) falls outside the feasible set \( OABC \), then any interior point is dominated by some point on the four sides of the feasible region: \( OA, CO, BC \) and \( AB \); therefore, the minimum of \( \nabla_{II} \left( Q_1, Q_2 \mid \xi_0, l, \bar{T} \right) \) could only be achieved at one of the four points \( Q^{OA}, Q^{CO}, Q^{BC} \) and \( Q^{AB} \). If \( Q^{UC} = (Q_1 (\xi_0), Q_2 (\xi_0)) \) is an interior point of the feasible set \( OABC \), then any interior point is dominated by either \( Q^{UC} \) or some point on the four sides. Thus, part 2) follows. ■
### Modeling parameters and their values for all the numerical examples

<table>
<thead>
<tr>
<th>Figure</th>
<th>Modeling parameters values</th>
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</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3, c_f - c_d = 2$</td>
</tr>
<tr>
<td>2.a, 2.b</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3$</td>
</tr>
<tr>
<td>3.</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3 \text{ or } 5, c_f - c_d = 2$</td>
</tr>
<tr>
<td>4.a, 4.b</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3 \text{ or } 5, c_f - c_d = 2$</td>
</tr>
<tr>
<td>6.a, 6.b</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3 \text{ or } 5, c_f - c_d = 2, c_f - c_n = 1.5$</td>
</tr>
<tr>
<td>7.a, 7.b</td>
<td>$\pi = 1.8, h = 0.3, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3, c_n - c_d = 1, l_f = 5$</td>
</tr>
<tr>
<td>8.</td>
<td>$\pi = 1.8, T = 14, \xi \sim \text{Gamma}(\mu, \theta), \mu = 5, \theta = 3, c_f = 5, c_n = 4, c_d = 3$</td>
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