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Appendix A. Proofs

We prove Theorem 2.1 for the case when $g(\tau)$ is strictly convex and differentiable through a series of lemmas. The general case follows similarly.

LEMMA A.1 *There exists a unique threshold $F_{M-1}(t)$ for the $(M-1)$st release, and the optimal cost function is given by*

$$V_{M-1}(B,t) = \begin{cases} K_M + K_{M-1} + c_{M-1}B & \text{if } B \geqslant F_{M-1}(t), \\ \quad +c_M EG(0, T_t^M) & \\ K_M + c_M EG(B, T_t^M) & \text{if } B < F_{M-1}(t). \end{cases} \tag{13}$$

PROOF. Since a final coordination is enforced, the optimal cost function for the last release is

$$V_M(B,t) = K_M + c_M B. \tag{14}$$

Substituting (14) into (7) and (8) with $m = M-1$ gives

$$V^c_{M-1}(B,t) = K_M + K_{M-1} + c_{M-1}B + c_M EG(0, T_t^M),$$
$$V^n_{M-1}(B,t) = K_M + c_M EG(B, T_t^M).$$

Then,

$$V^c_{M-1}(B,t) - V^n_{M-1}(B,t) \tag{15}$$
$$= K_{M-1} + c_M EG(0, T_t^M) - (c_M EG(B, T_t^M) - c_{M-1}B).$$

By assumption, the right-hand side of (15) is decreasing in B. Also,

$$V^c_{M-1}(0,t) - V^n_{M-1}(0,t) = K_{M-1} > 0.$$

Hence there is a unique $F_{M-1}(t)$ such that (15) equals zero at $B = F_{M-1}(t)$. This concludes the proof. □

LEMMA A.2 *$V^c_m(B,t) - V^n_m(B,t)$ is strictly decreasing in B.*

PROOF. We prove the result by induction. From the proof of Lemma A.1, the result is true for $m = M-1$. Suppose the result is true for $m+1$. Note that $V^c_{m+1}(0,t) - V^n_{m+1}(0,t) = K_{m+1} > 0$. Hence, the induction hypothesis would imply that there is a unique threshold $F_{m-1}(t)$ such that

$$V_{m+1}(B,t) = \begin{cases} V^n_{m+1}(B,t) & \text{if } B < F_{m+1}(t), \\ V^c_{m+1}(B,t) & \text{if } B \geqslant F_{m+1}(t). \end{cases}$$

Now we consider the cost function for the mth release. If coordination is carried out, the optimal cost is

$$V^c_m(B,t) K_m + c_m B + EV_{m+1}[G(0, T_t^{m+1}), t + T_t^{m+1}].$$

If no coordination is carried out, there are two cases to consider. Denote

$$\tilde{F}_{m+1}(t) = \max\{F_{m+1}(t+\tau) : \tau \in S_{T_t^{m+1}}\},$$

where $S_{T_t^{m+1}}$ is the support of T_t^{m+1}.

Case 1: If $B < \tilde{F}_{m+1}(t)$, then there is a nonempty set $\mathcal{T} = \{T | G(B,T) < F_{m+1}(t+T)\}$. Also define $\bar{\mathcal{T}} = \{T | G(B,T) \geqslant F_{m+1}(t+T)\}$. Thus, the optimal cost function is given by

$$V^n_m(B,t)$$
$$= \int_{\mathcal{T}} V^n_{m+1}[G(B,T), t+T] d\Phi_t^{m+1}(T)$$
$$+ \int_{\bar{\mathcal{T}}} V^c_{m+1}[G(B,T), t+T] d\Phi_t^{m+1}(T). \tag{16}$$

By induction hypothesis, we have

$$\frac{\partial V^n_{m+1}(B,t)}{\partial B} > \frac{\partial V^c_{m+1}(B,t)}{\partial B} = c_{m+1}.$$

Hence,

$$\frac{\partial V^n_m(B,t)}{\partial B}$$
$$= \int_{\mathcal{T}} \frac{\partial V^n_{m+1}[G(B,T), t+T]}{\partial G} \frac{\partial G(B,T)}{\partial B} d\Phi_t^{m+1}(T)$$
$$+ \int_{\bar{\mathcal{T}}} \frac{\partial V^c_{m+1}[G(B,T), t+T]}{\partial G} \frac{\partial G(B,T)}{\partial B} d\Phi_t^{m+1}(T)$$
$$> \int_0^\infty c_{m+1} \frac{\partial G(B,T)}{\partial B} d\Phi_t^{m+1}(T) > c_{m+1}.$$

Case 2: If $B \geqslant \tilde{F}_{m+1}$, then the coordination is carried out for sure at the $(m+1)$st release. The optimal cost function for not coordinating is

$$V^n_m(B,t) = \int_0^\infty V^c_{m+1}[G(B,T), t+T] d\Phi_t^{m+1}(T). \tag{17}$$

Then

$$\frac{\partial V^n_m(B,t)}{\partial B} = \int_0^\infty c_{m+1} \frac{\partial G(B,T)}{\partial B} d\Phi_t^{m+1}(T) > c_{m+1}.$$

Combine the two cases, we have

$$\frac{\partial V_m^c(B,t)}{\partial B} - \frac{\partial V_m^n(B,t)}{\partial B} < c_m - c_{m+1} < 0.$$

Hence, we conclude the lemma together with the fact that $V_m^c(0,t) - V_m^n(0,t) = K_m$. □

PROOFOF THEOREM 2.1. Follows directly from Lemma A.2. □

PROOFOF PROPOSITION 2.2. We first note that $G(B,T) = B + (B+k_c)(e^{\beta T} - 1)$ is linear in B. Also $EG(B,T) = B + (B+k_c)(q_t^m - 1)$. Substituting it into (13), we obtain

$$\begin{aligned} & V_{M-1}(B,t) \\ &= \begin{cases} K_M + K_{M-1} + c_{M-1}B & \text{if } B \geqslant F_{M-1}(t), \\ \quad +c_M(q_t^{M-1} - 1)k_c & \\ K_M + c_M q_t^{M-1} B & \text{if } B < F_{M-1}(t). \\ \quad +c_M(q_t^{M-1} - 1)k_c & \end{cases} \end{aligned} \tag{18}$$

Since $q_t^m = Ee^{\beta T_t^{m+1}} > 1$, we deduce that $V_{M-1}(B,t)$ is concave in B. Note also $F_{M-1}(t) = K_{M-1}/(c_M q_t^{M-1} - c_{M-1})$.

Now suppose that $V_{m+1}(B)$ is concave in B. Then, for $0 \leqslant \theta \leqslant 1$, we have

$$\begin{aligned} & \theta V_m^n(B_1,t) + (1-\theta)V_m^n(B_2,t) \\ &= \theta E\left[V_{m+1}\left(G(B_1, T_t^{m+1}), t + T_t^{m+1}\right)\right] \\ &\quad +(1-\theta)E\left[V_{m+1}\left(G(B_2, T_t^{m+1}), t + T_t^{m+1}\right)\right] \\ &\leqslant E[V_{m+1}(\theta G(B_1, T_t^{m+1}) \\ &\qquad\qquad +(1-\theta)G(B_2, T_t^{m+1}), t + T_t^{m+1})] \\ &= V_{m+1}^n\left(\theta B_1 + (1-\theta)B_2, t + T_t^{m+1}\right). \end{aligned}$$

Since $V_m^c(B,t)$ is linear, we conclude that $V_m(B,t)$ is concave in B.

To show $V_m^n(B,t)$ is smooth, we only need to show that $\partial V^n/\partial B|_{B \to \tilde{F}^-} = q_t^m$. This is established by differentiating (16) with respect to B. □

PROOFPROPOSITION 2.3. Let $B^0 = 0$ and $B^k = \sum_{j=1}^k \alpha_j \delta$ be the kth break point on $g(t)$. Also write $T = n\delta + \xi$, where n is a nonnegative integer and $0 \leqslant \xi < \delta$. Define $B_\delta^k = B^k - \alpha_k \xi$. Then

$$\begin{aligned} G(B^k, T) &= B^{k+n} + \alpha_{k+n+1}\xi \\ G(B_\delta^k, T) &= B^{k+n}. \end{aligned}$$

Note that $G(B,T)$ is piecewise linear in B with the break points $\{B^0, B_\delta^1, B^1, B_\delta^2, ...\}$. We examine the slopes of consecutive pieces of $G(B,T)$:

$$\begin{aligned} & \frac{G(B^k,T) - G(B_\delta^k,T)}{B^k - B_\delta^k} \\ &= \frac{B^{k+n} + \alpha_{k+n+1}T - B^{k+n}}{B^k - B^k + \alpha_{k+1}T} \\ &= \frac{\alpha_{k+n+1}}{\alpha_{k+1}} \end{aligned}$$

and

$$\begin{aligned} & \frac{G(B_\delta^k,T) - G(B^{k-1},T)}{B_\delta^k - B^{k-1}} \\ &= \frac{B^{k+n} - B^{k+n-1} - \alpha_{k+n}T}{B^k - \alpha_k T - B^{k-1}} \\ &= \frac{\alpha_{k+n}}{\alpha_k}. \end{aligned}$$

Clearly, the slopes of $G(B^k,T)$ is monotone increasing (decreasing) if α_{k+1}/α_k is monotone increasing (decreasing). Note also that T is arbitrary. Hence, we conclude the proof. □

PROOFOF PROPOSITION 2.4. It follows from an inductive argument and the fact that the composition of an increasing concave function to a concave function is concave. □

PROOFOF THEOREM 2.2. We prove the result for the case when $G_m^i(B,T)$ is differentiable in B. At the mth release, define $\tau(\bar{B})$ to be the solution of (10).

After T_t^{m+1} time units, the $(m+1)$st module is released, and the effective fault level becomes

$$\bar{G}_{m+1}(\bar{B}, T_t^{m+1}) = \sum_{i=1}^k \alpha_{m+1}^i G^i(0, \tau(\bar{B}) + T_t^{m+1}).$$

The function $\bar{G}_{m+1}(\bar{B}, T_t^{m+1})$ represents the effective fault level at the $(m+1)$st release given that the effective fault level at the mth release is $\bar{B}$. Note that the growth of the effective fault level depends on the release index m due to its dependence on c_m^i.

From the proof of Theorem 2.1, the optimality of a threshold policy can be established if we show that

$$\bar{c}_{m+1} E \frac{\partial \bar{G}_{m+1}(\bar{B}, T_t^{m+1})}{d\bar{B}} - \bar{c}_m > 0.$$

Differentiating (10) with respect to $\bar{B}$ and solving for $d\tau/d\bar{B}$, we obtain

$$\frac{d\tau}{d\bar{B}} = \frac{1}{\sum_{i=1}^k \alpha_m^i \frac{\partial G^i(0,\tau)}{\partial \tau}} = \frac{\bar{c}_m}{\sum_{i=1}^k c_m^i \frac{\partial G^i(0,\tau)}{\partial \tau}}.$$

Also, since $G^i(B,T) - B$ increases in B, it follows directly that $\partial G^i(0,\tau)/\partial\tau$ increases in τ. Hence, we have

$$\begin{aligned}
&\bar{c}_{m+1} E \frac{\partial \bar{G}_{m+1}(\bar{B}, T_t^{m+1})}{\partial \bar{B}} \\
&= \bar{c}_{m+1} E \frac{\partial \sum_{i=1}^k \alpha_{m+1}^i G^i(0, \tau(\bar{B}) + T_t^{m+1})}{\partial \bar{B}} \\
&= \sum_{i=1}^k E \frac{\partial c_{m+1}^i G^i(0,s)}{\partial s}\Big|_{s=\tau(\bar{B})+T_t^{m+1}} \\
&\qquad \cdot \frac{\bar{c}_m}{\sum_{i=1}^k c_m^i \frac{\partial G^i(0,\tau)}{\partial \tau}|_{\tau=\tau(\bar{B})}} \\
&> \bar{c}_m.
\end{aligned}$$

Thus, we conclude the proof. □

PROOFOF PROPOSITION 3.1. From Lemma A.1, the threshold F_{M-1} satisfies

$$c[EG(B,T) - B] = K + CEG(0,T).$$

At the mth release, suppose $F_m > F_{M-1}$. Then we have

$$\begin{aligned}
&V_m^c(F_{M-1}) - V_m^n(F_{M-1}) \\
&= K + cF_{M-1} + V_m(0) \\
&\qquad -[K + cEG(F_{M-1}, T) + V_{m+1}(0)] \\
&= V_m(0) - c[EG(F_{M-1}, T) - F_{M-1}] - V_{m+1}(0) \\
&= V_m(0) - [K + cEG(0,T) + V_{m+1}(0)] \\
&\leqslant 0,
\end{aligned}$$

which indicates that coordination should be carried out at $B = F_{M-1}$. Thus, we have a contradiction, which proves the lemma. □

PROOFOF PROPOSITION 3.2. In the reverse-time setting, let $W_l(B) = V_m(B) - V_m(0)$ with $V_m(\cdot)$ defined in (5), (7) and (8). Then

$$W_{l+1}(B) = \min\{EW_l^n(G(B,T)), K + cB\}, \quad (19)$$

and

$$\begin{aligned}
W_l^n(B) &= \int_0^{T_{l-1}(B)} W_{l-1}^n(G(B,T))\, d\Phi(T) \\
&\quad - \int_0^{T_{l-1}(0)} W_{l-1}^n(G(0,T))\, d\Phi(t) \\
&\quad + \int_{T_{l-1}(B)}^{\infty} [K + cG(B,T)] d\Phi(T) \\
&\quad - \int_{T_{l-1}(0)}^{\infty} [K + cG(0,T)] d\Phi(T),
\end{aligned}$$

where $T_l(B)$ solves $G(B,T_l) = f_l$ and f_l is the optimal threshold value in Theorem 2.1.

Clearly, if the value functions $W_l(\cdot)$ converge, then the threshold values must also converge. Thus, we try to show that $W_l^n(\cdot)$ converges in l, which, in turn implies the convergence of $W_l(\cdot)$.

Let S denote the collection of all the value functions W_l^n, $l > 2$, in the dynamic system. We first note that every element in S is continuously differentiable. It follows that S is *equicontinuous*. That is, for any $\delta > 0$, there is a $\varepsilon > 0$, such that $|w(x) - w(y)| < \delta$ for $y \in (x-\varepsilon, x+\varepsilon)$ and each $w \in S$. Clearly, S is *point-wise bounded*. That is, $w(x) \leqslant K + cx < \infty$ for each $x \in [0,\infty)$ and each $w \in S$. Thus, by the Arzela-Ascoli theorem(page 245 of Rudin 1987), there is a subsequence of $\{w_l^n\}$ that converges uniformly as $k_l \to \infty$. Hence, we conclude the proposition. □

Proposition A.1 *Suppose that the inter-release times are independent. Let P_m, $1 \leqslant m \leqslant M$, be the probability of coordinating at the mth release. Also, let $P_0 = 1$. Then $P_M = 1$ and*

$$P_m = \sum_{i=0}^{m-1} P_{m|i} P_i (1 - P_{i+1|i}) \ldots (1 - P_{m-1|i}) \quad (20)$$

$$\textit{for } 1 \leqslant m \leqslant M-1, \quad (21)$$

where

$$P_{m|i} = \int_{\frac{1}{\beta} ln \frac{F_m + k_c}{k_c}}^{\infty} d\Psi(t) \quad (22)$$

and $\Psi = \Phi^i * \ldots * \Phi^m$ is the convolution of $\Phi^i, \ldots, \Phi^m$.

PROOF. Let B_m be the number of faults at the end of the mth release. Also denote t_m as the stopping time to the mth release. Under the optimal threshold policy, the probability of coordinating at release 1 is given by

$$\begin{aligned}
P_1 &= P\{B_1 \geqslant F_1\} = P\{k_c(e^{\beta t_1} - 1) > F_1\} \\
&= \int_{\frac{1}{\beta} ln \frac{F_1 + k_c}{k_c}}^{\infty} d\Phi^1(t). \quad (23)
\end{aligned}$$

If the last coordination is at the ith release, then the random variable $t_m - t_i$, $i < m$, has the distribution $\Phi^i * \ldots * \Phi^m$. Thus, the conditional probability of coordinating at the mth release is given by

$$\begin{aligned}
&P_{m|i} \\
&= P\{B_m \geqslant F_m | B_i \geqslant F_i; B_k < F_k, i < k < m\} \\
&= P\{k_c(e^{\beta(t_m - t_i)} - 1) \geqslant F_m | B_i \geqslant F_i; B_k < F_k, i < k < m\} \\
&= \int_{\frac{1}{\beta} ln \frac{F_m + k_c}{k_c}}^{\infty} d\Phi^i * \ldots * \Phi^m(t).
\end{aligned}$$

Using the conditioning rule, we have

$$
\begin{aligned}
& P\{B_i \geqslant F_i; B_k < F_k, i < k < m\} \\
= & P\{B_{m-1} < F_{m-1} | B_i \geqslant F_i; B_k < F_k, i < k < m-1\} \\
& \cdot P\{B_i \geqslant F_i; B_k < F_k, i < k < m-1\} \\
= & P\{B_{m-1} < F_{m-1} | B_i \geqslant F_i; B_k < F_k, i < k < m-1\} \\
& ... P\{B_{i+1} < F_{i+1} | B_i \geqslant F_i\} P\{B_i \geqslant F_i\} \\
= & (1 - P_{m-1|i}) ... (1 - P_{i+1|i}) P_i.
\end{aligned}
$$

Thus, we can calculate P_m by unconditioning to yield (20). □

Appendix B. The Release-Based Policy.

Let B_m to be the number of faults when development has been continued without coordination for m releases. Denote $C(m,k)$ as the coordination cost at the kth release when the last coordination is at the mth release. Then $C(m,k) = K + bk + cB_{k-m}$. Also define an array $u[m]$ to be the optimal cost from the first coordination after the mth release to the end of the project and an array $t[m]$ to be the index of the next coordination if we coordinate at the mth release. The algorithm proceeds as follows:

1 Initialize: $u[M] = 0$.

2 For $m = M$ and $m \geqslant 0$,

 2.1 Set $u[m] = C(m, m+1) + u[m+1]$ and $t[m] = m+1$.

 2.2 For $k = m+2$ and $k < M$, if $C(m,k) + u[k] < u[m]$, we set $u[m] = C(m,k) + u[k]$ and $t[m] = k$.

 2.3 $m = m + 1$.

At the end of the algorithm, $u[0]$ returns the total coordination cost.