Appendix: Proofs

Proof of Theorem 1:

By induction: Equation (5) establishes the base of the induction for n=0. Note that (4) is satisfied by the construction of A. Suppose that the hypothesis is true for all values less than k. From (7)

$$v_{k+1}'(t) = -a(t)\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(v_{k+1}(t)-v_k(t)\right)^{1-\varepsilon} = -a(t)\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(v_{k+1}(t)-\beta_k(A(t))^{1/\varepsilon}\right)^{1-\varepsilon}$$

This is a linear ordinary differential equation, so we need only verify that the solution holds:

$$v_{k+1}'(t)+a(t)\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(v_{k+1}(t)-\beta_k(A(t))^{1/\varepsilon}\right)^{1-\varepsilon}$$

$$=\beta_{k+1}\frac{1}{\varepsilon}(A(t))^{1/\varepsilon-1}A'(t)+a(t)\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(\beta_{k+1}(A(t))^{1/\varepsilon}-\beta_k(A(t))^{1/\varepsilon}\right)^{1-\varepsilon}$$

$$=-\beta_{k+1}\frac{1}{\varepsilon}(A(t))^{\frac{1-\varepsilon}{\varepsilon}}a(t)+a(t)\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}(\beta_{k+1}-\beta_k)^{1-\varepsilon}(A(t))^{\frac{1-\varepsilon}{\varepsilon}}$$

$$=(A(t))^{\frac{1-\varepsilon}{\varepsilon}}a(t)\left(-\beta_{k+1}\frac{1}{\varepsilon}+\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}(\beta_{k+1}-\beta_k)^{1-\varepsilon}\right)=0,$$

which establishes the hypothesis at k+1 as desired.

Given the formula for v_n, the price posted satisfies

$$p_n(t)=\frac{\varepsilon}{\varepsilon-1}(v_n(t)-v_{n-1}(t))=\frac{\varepsilon}{\varepsilon-1}A(t)^{1/\varepsilon}(\beta_n-\beta_{n-1})=\beta_n^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}$$

since $\beta_n-\beta_{n-1}=\frac{\varepsilon-1}{\varepsilon}\beta_n^{-1/(\varepsilon-1)}$. Q.E.D.

Proof of Theorem 2:

Define $\gamma_n = \frac{\beta_n}{n^{\frac{\varepsilon-1}{\varepsilon}}}$. The theorem states that γ_n converges to 1. Using (8),

we have

$$\gamma_n n^{\frac{\varepsilon-1}{\varepsilon}}\left(\gamma_n n^{\frac{\varepsilon-1}{\varepsilon}} - \gamma_{n-1}(n-1)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\varepsilon-1} = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1}$$

or

$$\gamma_n^{\frac{1}{\varepsilon-1}} n^{\frac{1}{\varepsilon}}\left(\gamma_n n^{\frac{\varepsilon-1}{\varepsilon}} - \gamma_{n-1}(n-1)^{\frac{\varepsilon-1}{\varepsilon}}\right) = \frac{\varepsilon-1}{\varepsilon}$$

or

$$\gamma_n^{\frac{1}{\varepsilon-1}} n\left(\gamma_n - \gamma_{n-1}\left(\frac{n-1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) = \frac{\varepsilon-1}{\varepsilon}$$

Claim 1: $\gamma_n \leq 1$.

Proof of Claim 1: Note that $\gamma_1 = \beta_1 = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\varepsilon-1}{\varepsilon}} < 1$. Suppose, by way of contradiction, that γ_m is the first instance of $\gamma_m > 1$. Then $\gamma_m > 1 \geq \gamma_{m-1}$. Thus

$$\frac{\varepsilon-1}{\varepsilon} = \gamma_m^{\frac{1}{\varepsilon-1}} m\left(\gamma_m - \gamma_{m-1}\left(\frac{m-1}{m}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) \geq \gamma_m^{\frac{1}{\varepsilon-1}} m\left(\gamma_m - \gamma_m\left(\frac{m-1}{m}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)$$

$$= \gamma_m^{\frac{\varepsilon}{\varepsilon-1}} m\left(1 - \left(\frac{m-1}{m}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) \geq \gamma_m^{\frac{\varepsilon}{\varepsilon-1}} \frac{\varepsilon-1}{\varepsilon},$$

since $m\left(1-\left(\frac{m-1}{m}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)$ is a decreasing sequence that converges to $\frac{\varepsilon-1}{\varepsilon}$. This verifies claim 1.

Now rewrite

$$\gamma_n^{\frac{1}{\varepsilon-1}} n\left(\gamma_n - \gamma_{n-1}\left(\frac{n-1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) = \frac{\varepsilon-1}{\varepsilon}$$

to obtain

$$\gamma_n = \frac{\varepsilon-1}{n\varepsilon}\gamma_n^{\frac{-1}{\varepsilon-1}} + \gamma_{n-1}\left(\frac{n-1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}} \geq \frac{\varepsilon-1}{n\varepsilon} + \gamma_{n-1}\left(\frac{n-1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}},$$

with the inequality implied by claim 1.

Equality in this expression defines a new sequence η_n which is a lower bound for γ_n.

$$\eta_n = \frac{\varepsilon-1}{n\varepsilon} + \eta_{n-1}\left(\frac{n-1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}.$$

It is readily verified by induction that

$$\eta_n = \left(\frac{1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\eta_0 + \frac{\varepsilon-1}{\varepsilon}\sum_{j=1}^{n}\frac{1}{j}\left(\frac{j}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

$$= \left(\frac{1}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\eta_0 + \frac{\varepsilon-1}{\varepsilon}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^{\frac{-1}{\varepsilon}} \to \frac{\varepsilon-1}{\varepsilon}\int_0^1 x^{-1/\varepsilon}\,dx = 1.$$

Thus, γ_n is bounded between η_n and 1 and thus converges to 1.

From (9): $p_n(t) = \beta_n^{\frac{-1}{\varepsilon-1}}(A(t))^{1/\varepsilon} \approx \left(\frac{A(t)}{n}\right)^{1/\varepsilon}$.

The evolution of the probability that there are *n* items available at time *t* is governed by the differential equation

$$q_n'(t) = \lambda(p_{n+1}(t),t)q_{n+1}(t) - \lambda(p_n(t),t)q_n(t)$$

$$= a(t)(p_{n+1}(t))^{-\varepsilon} q_{n+1}(t) - a(t)(p_n(t))^{-\varepsilon} q_n(t)$$

$$= a(t)\left(\beta_{n+1}^{\varepsilon/\varepsilon-1} A(t)^{-1} q_{n+1}(t) - \beta_n^{\varepsilon/\varepsilon-1} A(t)^{-1} q_n(t)\right)$$

$$= \frac{a(t)}{A(t)}\left(\beta_{n+1}^{\varepsilon/\varepsilon-1} q_{n+1}(t) - \beta_n^{\varepsilon/\varepsilon-1} q_n(t)\right)$$

because q_n increases when a sale is made starting with n+1 items, and is decreased when a sale is made when n items remain. If the firm begins with N units at time 0, then $q(N,0)=1$ and $q(n,0)=0$ for all $n<N$.

Using the approximation, this becomes

$$q_n'(t) \approx \frac{a(t)}{A(t)}\left((n+1)q_{n+1}(t) - nq_n(t)\right),$$

which has the elegant binomial solution:

$$q_n(t) \approx \binom{N}{n}\left(\frac{A(t)}{A(0)}\right)^n\left(1-\frac{A(t)}{A(0)}\right)^{N-n}.$$

Q.E.D.

Proof of Theorem 4:

The expected value of the amount of remaining capacity, n is approximately $n \approx \frac{NA(t)}{A(0)}$. Inequality (17) is equivalent to this holding for all t, but it is more convenient to express it in terms of n, with $A(t) \approx \frac{nA(0)}{N}$. Then (17) can be expressed as $\frac{n-1}{n}\left(1+\frac{N}{\varepsilon n A(0)}\right)^{\varepsilon} \le 1$.

Let $\kappa(x) = (1-x)\left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon}$; It is sufficient to prove that $\kappa\left(1/n\right) \le 1$ for all n in [1,N].

$$\kappa'(x) = -\left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon} + (1-x)\left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon-1}\frac{N}{A(0)}$$

$$= \left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon-1}\left[-\left(1+\frac{N}{\varepsilon A(0)}x\right) + (1-x)\frac{N}{A(0)}\right]$$

$$= \left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon-1}\left[\frac{N}{A(0)} - 1 - x\frac{N}{A(0)}\left(1+\frac{1}{\varepsilon}\right)\right]$$

$$= \frac{1}{A(0)}\left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon-1}\left[N - A(0) - Nx\left(1+\frac{1}{\varepsilon}\right)\right]$$

$$\leq \frac{1}{A(0)}\left(1+\frac{N}{\varepsilon A(0)}x\right)^{\varepsilon-1}\left[N - A(0) - \left(1+\frac{1}{\varepsilon}\right)\right] \leq 0\,.$$

Thus, $\kappa\left(1/n\right) \leq \kappa\left(1/N\right) = \left(1 - 1/N\right)\left(1+\frac{1}{\varepsilon A(0)}\right)^{\varepsilon} \leq \left(1 - 1/A(0)\right)\left(1+\frac{1}{\varepsilon A(0)}\right)^{\varepsilon} \leq 1\,.$

The first inequality follows from $n \leq N$ and the fact that κ was shown to be decreasing; the second inequality from the hypothesis of the theorem that $N \leq A(0)$, and the third inequality by noting that $(1-z)\left(1+\frac{z}{\varepsilon}\right)^{\varepsilon}$ is a decreasing function of z, and thus maximized at z=0, so that

$(1-z)\left(1+\frac{z}{\varepsilon}\right)^{\varepsilon} \leq 1\,.$ Q.E.D.