Appendix: Proofs

Proof of Theorem 1:

By induction: Equation (5) establishes the base of the induction for $n=0$. Note that (4) is satisfied by the construction of $A$. Suppose that the hypothesis is true for all values less than $k$. From (7)

$$v_{k+1}'(t) = -a(t) \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} \left( v_{k+1}(t) - v_k(t) \right)^{1-\varepsilon} = -a(t) \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} \left( v_{k+1}(t) - \beta_k(A(t))^{1/\varepsilon} \right)^{1-\varepsilon}$$

This is a linear ordinary differential equation, so we need only verify that the solution holds:

$$v_{k+1}'(t) + a(t) \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} \left( v_{k+1}(t) - \beta_k(A(t))^{1/\varepsilon} \right)^{1-\varepsilon} = \beta_{k+1} \frac{1}{\varepsilon} (A(t))^{1/\varepsilon - 1} A'(t) + a(t) \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} \left( \beta_{k+1}(A(t))^{1/\varepsilon} - \beta_k(A(t))^{1/\varepsilon} \right)^{1-\varepsilon}$$

$$= -\beta_{k+1} \frac{1}{\varepsilon} (A(t))^{1-\varepsilon} a(t) + a(t) \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} \left( \beta_{k+1} - \beta_k \right)^{1-\varepsilon} (A(t))^{1-\varepsilon}$$

$$= (A(t))^{1-\varepsilon} a(t) \left( -\beta_{k+1} \frac{1}{\varepsilon} + \frac{(\varepsilon - 1)^{1-\varepsilon}}{\varepsilon} (\beta_{k+1} - \beta_k)^{1-\varepsilon} \right) = 0,$$

which establishes the hypothesis at $k+1$ as desired.

Given the formula for $v_n$, the price posted satisfies

$$p_n(t) = \frac{\varepsilon}{\varepsilon - 1} (v_n(t) - v_{n-1}(t)) = \frac{\varepsilon}{\varepsilon - 1} A(t)^{1/\varepsilon} (\beta_n - \beta_{n-1}) = \beta_n^{-1/\varepsilon - 1} A(t)^{1/\varepsilon}$$

since $\beta_n - \beta_{n-1} = \frac{\varepsilon - 1}{\varepsilon} \beta_n^{-1/\varepsilon - 1}$. Q.E.D.
Proof of Theorem 2:

Define $\gamma_n = \frac{\beta_n}{n^{\varepsilon-1}}$. The theorem states that $\gamma_n$ converges to 1. Using (8), we have

$$
\gamma_n n^{\varepsilon-1} \left( \gamma_n^{\varepsilon-1} - \gamma_{n-1}^{\varepsilon-1} (n-1)^{\varepsilon-1} \right) = \left( \frac{\varepsilon-1}{\varepsilon} \right)^{\varepsilon-1}
$$

or

$$
\frac{1}{\gamma_n^{\varepsilon-1}} n^{\varepsilon-1} \left( \gamma_n^{\varepsilon-1} - \gamma_{n-1}^{\varepsilon-1} (n-1)^{\varepsilon-1} \right) = \frac{\varepsilon-1}{\varepsilon}
$$

or

$$
\frac{1}{\gamma_n^{\varepsilon-1}} n \left( \gamma_n - \gamma_{n-1} \left( \frac{n-1}{n} \right)^{\varepsilon-1} \right) = \frac{\varepsilon-1}{\varepsilon}
$$

Claim 1: $\gamma_n \leq 1$.

Proof of Claim 1: Note that $\gamma_1 = \frac{\varepsilon-1}{\varepsilon} < 1$. Suppose, by way of contradiction, that $\gamma_m$ is the first instance of $\gamma_m > 1$. Then $\gamma_m \geq \gamma_{m-1}$. Thus

$$
\frac{\varepsilon-1}{\varepsilon} = \frac{1}{\gamma_m^{\varepsilon-1}} m \left( \gamma_m - \gamma_{m-1} \left( \frac{m-1}{m} \right)^{\varepsilon-1} \right) \geq \frac{1}{\gamma_m^{\varepsilon-1}} m \left( \gamma_m - \gamma_m \left( \frac{m-1}{m} \right)^{\varepsilon-1} \right)
$$

$$
= \gamma_m^{\varepsilon-1} m \left( 1 - \left( \frac{m-1}{m} \right)^{\varepsilon-1} \right) \geq \gamma_m^{\varepsilon-1} \frac{\varepsilon-1}{\varepsilon},
$$
since \( m \left( 1 - \left( \frac{m-1}{m} \right) \frac{\varepsilon^{-1}}{\varepsilon} \right) \) is a decreasing sequence that converges to \( \frac{\varepsilon-1}{\varepsilon} \). This verifies claim 1.

Now rewrite

\[
\frac{1}{\gamma_n^{-1}} n \left( \gamma_n - \gamma_{n-1} \left( \frac{n-1}{n} \right) \frac{\varepsilon^{-1}}{\varepsilon} \right) = \frac{\varepsilon-1}{\varepsilon}
\]

to obtain

\[
\gamma_n = \frac{\varepsilon-1}{n\varepsilon} \gamma_{n-1}^{-1} + \gamma_{n-1} \left( \frac{n-1}{n} \right) \frac{\varepsilon^{-1}}{\varepsilon} \geq \frac{\varepsilon-1}{n\varepsilon} + \gamma_{n-1} \left( \frac{n-1}{n} \right) \frac{\varepsilon^{-1}}{\varepsilon},
\]

with the inequality implied by claim 1.

Equality in this expression defines a new sequence \( \eta_n \) which is a lower bound for \( \gamma_n \).

\[
\eta_n = \frac{\varepsilon-1}{n\varepsilon} + \eta_{n-1} \left( \frac{n-1}{n} \right) \frac{\varepsilon^{-1}}{\varepsilon}.
\]

It is readily verified by induction that

\[
\eta_n = \left( \frac{1}{n} \right)^{\varepsilon^{-1}} \eta_0 + \frac{\varepsilon^{-1}}{\varepsilon} \sum_{j=1}^{n} \left( \frac{j}{n} \right)^{\varepsilon^{-1}}
\]

\[
= \left( \frac{1}{n} \right)^{\varepsilon^{-1}} \eta_0 + \frac{\varepsilon^{-1}}{\varepsilon} \sum_{j=1}^{n} \left( \frac{j}{n} \right)^{-1} \rightarrow \frac{\varepsilon-1}{\varepsilon} \int_{0}^{1} x^{-1/\varepsilon} dx = 1.
\]

Thus, \( \gamma_n \) is bounded between \( \eta_n \) and 1 and thus converges to 1.

From (9): \( p_n(t) = \beta_n^{-1} (A(t))^{-1/\varepsilon} \approx \left( \frac{A(t)}{n} \right)^{1/\varepsilon} \).
The evolution of the probability that there are \( n \) items available at time \( t \) is governed by the differential equation

\[
q'_n(t) = \lambda(p_{n+1}(t), t)q_{n+1}(t) - \lambda(p_n(t), t)q_n(t) = a(t)(p_{n+1}(t))^{-\epsilon}q_{n+1}(t) - a(t)(p_n(t))^{-\epsilon}q_n(t)
\]

\[
= a(t)\left(\beta_n^{\epsilon/\epsilon - 1}A(t)^{-1}q_{n+1}(t) - \beta_n^{\epsilon/\epsilon - 1}A(t)^{-1}q_n(t)\right)
\]

\[
= \frac{a(t)}{A(t)}\left(\beta_n^{\epsilon/\epsilon - 1}q_{n+1}(t) - \beta_n^{\epsilon/\epsilon - 1}q_n(t)\right)
\]

because \( q_n \) increases when a sale is made starting with \( n+1 \) items, and is decreased when a sale is made when \( n \) items remain. If the firm begins with \( N \) units at time 0, then \( q(N,0)=1 \) and \( q(n,0)=0 \) for all \( n<N \).

Using the approximation, this becomes

\[
q'_n(t) \approx \frac{a(t)}{A(t)}((n+1)q_{n+1}(t) - nq_n(t)),
\]

which has the elegant binomial solution:

\[
q_n(t) \approx \left(\frac{N}{n}\right)^n\left(\frac{A(t)}{A(0)}\right)^n\left(1 - \frac{A(t)}{A(0)}\right)^{N-n}.
\]

Q.E.D.

Proof of Theorem 4:

The expected value of the amount of remaining capacity, \( n \) is approximately \( n \approx \frac{NA(t)}{A(0)} \).

Inequality (17) is equivalent to this holding for all \( t \), but it is more convenient to express it in terms of \( n \), with \( A(t) \approx \frac{nA(0)}{N} \). Then (17) can be expressed as \( \frac{n-1}{n} \left(1 + \frac{N}{\epsilon n A(0)}\right)^x \leq 1 \).

Let \( \kappa(x) = (1 - x)^x \left(1 + \frac{N}{\epsilon A(0)}x\right)^x \); It is sufficient to prove that \( \kappa\left(\frac{1}{n}\right) \leq 1 \) for all \( n \) in \([1, N]\).
\[ \kappa'(x) = \left(1 + \frac{N}{\varepsilon A(0)} x\right)^\varepsilon + (1 - x) \left(1 + \frac{N}{\varepsilon A(0)} x\right)^{\varepsilon-1} \frac{N}{A(0)} \]

\[ = \left(1 + \frac{N}{\varepsilon A(0)} x\right)^{\varepsilon-1} \left[ - \left(1 + \frac{N}{\varepsilon A(0)} x\right) + (1 - x) \frac{N}{A(0)} \right] \]

\[ = \left(1 + \frac{N}{\varepsilon A(0)} x\right)^{\varepsilon-1} \left[ \frac{N}{A(0)} - 1 - x \frac{N}{A(0)} \left(1 + \frac{1}{\varepsilon}\right) \right] \]

\[ = \frac{1}{A(0)} \left(1 + \frac{N}{\varepsilon A(0)} x\right)^{\varepsilon-1} \left[ N - A(0) - Nx \left(1 + \frac{1}{\varepsilon}\right) \right] \]

\[ \leq \frac{1}{A(0)} \left(1 + \frac{N}{\varepsilon A(0)} x\right)^{\varepsilon-1} \left[ N - A(0) - \left(1 + \frac{1}{\varepsilon}\right) \right] \leq 0. \]

Thus, \( \kappa \left(\frac{1}{n}\right) \leq \kappa \left(\frac{1}{N}\right) = \left(1 - \frac{1}{N} \right) \left(1 + \frac{1}{\varepsilon A(0)} \right)^\varepsilon \leq \left(1 - \frac{1}{A(0)} \right) \left(1 + \frac{1}{\varepsilon A(0)} \right)^\varepsilon \leq 1 \).

The first inequality follows from \( n \leq N \) and the fact that \( \kappa \) was shown to be decreasing; the second inequality from the hypothesis of the theorem that \( N \leq A(0) \), and the third inequality by noting that \( (1 - z) \left(1 + \frac{z}{\varepsilon}\right)^\varepsilon \) is a decreasing function of \( z \), and thus maximized at \( z = 0 \), so that

\[ (1 - z) \left(1 + \frac{z}{\varepsilon}\right)^\varepsilon \leq 1. \] Q.E.D.