Appendix A: Proof of Theorem 1

We first consider \( k \) even; the result for \( k \) odd easily follows. Let \( u_{j} \) denote the number of sequences \( \sigma \) corresponding to Case \( j \) that occur in a \( k \)-unit cycle, for all \( j \) and \( r \) \((j = 2, 3, \ldots, 9, \text{and } r = 1, \ldots, \frac{k}{2}) \) and machines \( M_{i}, i = 1, \ldots, m \). As there are \( \frac{mk}{2} \) sequences for each machine in a \( k \)-unit cycle, we have

\[
mk = u_{2} + u_{3} + u_{4} + u_{5} + u_{6} + u_{7} + u_{8} + u_{9}.
\]

Let \( U_{j} \) be the collection of indices of machines in Case \( j \). If a machine \( M_{i} \) has \( s \) sequences in Case \( j \), then there will be \( s \) instances of \( i \) in \( U_{j} \). \( u_{0} \) (respectively, \( u_{1} \)) represents the number of visits to \( I \) (\( O \)) during which the robot unloads (loads) two parts.

By adding residence times corresponding to the possible cases for a machine in a dual-gripper robot cell, we get a lower bound for \( T_{r} \), the aggregate residence time of the robot at all machines for a \( k \)-unit cycle:

\[
T_{r} \geq 2(m + 1)k\epsilon + (u_{0} + u_{1} + u_{4} + u_{5} + u_{7} + 2u_{8} + u_{9})\theta + \sum_{i \in U_{3}} p_{i} + \sum_{i \in U_{5}} p_{i} + 2 \sum_{i \in U_{6}} p_{i} + \sum_{i \in U_{7}} p_{i} + 2 \sum_{i \in U_{8}} p_{i}.
\]

The computation of required movement time is identical to that of Geismar et al. (2006):

\[
T_{t} = (m + 2)k\delta + (-u_{0} - u_{1} + 2u_{2} + u_{3} + u_{4} - u_{9})\delta.
\]

Hence, we have lower bound \( T_{r} + T_{t} \), which can be considered as a fixed amount ((\( m + 2)k\delta + 2(m + 1)k\epsilon \)) plus an amount that varies according to which case each sequence is assigned. This is represented for machines \( M_{1}, \ldots, M_{m} \) by the summation term in expression (1). Table 4 lists the minimum time added to the cycle time for each 2-unit sequence that is assigned to a particular case. The last term in expression (1) corresponds to the time added to the per unit cycle time if \( u_{0} = u_{1} = 1 \).

The added time for Case 5 or Case 7 is not included in the summation term because \( p_{i} + \theta \geq \min\{2p_{i}, 2\theta\} \). Similarly, the added times for Cases 3 and 4 are not included because \( p_{i} + \delta \geq \min\{2p_{i}, 2\delta\} \) and \( \delta + \theta \geq \min\{2\delta, 2\theta\} \), respectively.

For \( k \) odd, a lower bound can be found by considering the cycle to be the concatenation of \( (k - 1)/2 \) subsequences \((M_{1,2r-1}^{l}, \sigma_{1}, M_{1,2r-1}^{u}, \sigma_{2}, M_{1,2r}^{l}, \sigma_{3}, M_{1,2r}^{u}, \sigma_{4})\) and a subsequence \((M_{1,k}^{l}, \sigma_{1}, M_{1,k}^{u}, \sigma_{2})\). Note that a lower bound for robot actions in a subsequence \((M_{1,k}^{l}, \sigma_{1}, M_{1,k}^{u}, \sigma_{2})\)
Table 4: Variable amount added to the cycle time for each case.

<table>
<thead>
<tr>
<th>Case</th>
<th>Added time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 0, ( u_0 = 1 )</td>
<td>( \theta - \delta )</td>
</tr>
<tr>
<td>Case 1, ( u_1 = 1 )</td>
<td>( \theta - \delta )</td>
</tr>
<tr>
<td>Case 2</td>
<td>( 2\delta )</td>
</tr>
<tr>
<td>Case 3</td>
<td>( p_i + \delta )</td>
</tr>
<tr>
<td>Case 4</td>
<td>( \delta + \theta )</td>
</tr>
<tr>
<td>Case 5</td>
<td>( p_i + \theta )</td>
</tr>
<tr>
<td>Case 6</td>
<td>( 2p_i )</td>
</tr>
<tr>
<td>Case 7</td>
<td>( p_i + \theta )</td>
</tr>
<tr>
<td>Case 8</td>
<td>( 2\theta )</td>
</tr>
<tr>
<td>Case 9</td>
<td>( 2p_i + \theta - \delta )</td>
</tr>
</tbody>
</table>

is \((m + 2)\delta + 2(m + 1)\epsilon + \sum_{i=1}^{m} \min\{p_i, \delta, \theta\}\), which is greater than or equal to the right-hand side of (1). Therefore, (1) is a lower bound for the per unit cycle time for \( k \) odd, too.

Appendix B: Proofs of Lemmas 1 through 4

Proof of Lemma 1:

a) If \( \max p_i \geq (m + 2)\delta + 2m\epsilon + (m - 1)\theta \), then \( S_m^2 = p_h + 2\epsilon + \theta \), which is a lower bound on the per unit cycle time of a dual-gripper robot cell (Geismar et al. 2006). The condition \( \theta \leq 3\delta + 2\epsilon \) ensures that \( p_h + 2\epsilon + \theta \leq p_h + 3\delta + 4\epsilon \), which is a lower bound for the per unit cycle time of a single-gripper robot cell (Dawande et al. 2002).

b) \( p_h \geq (m + 2)\delta + 2m\epsilon + (m - 1)\theta \) and \( \theta \geq 3\delta + 2\epsilon \) imply that \( \pi_D = p_h + 3\delta + 4\epsilon \leq p_h + 2\epsilon + \theta \), so \( \pi_D \) is optimal over all single-gripper and dual-gripper cycles.

Proof of Lemma 2:

a) \( \Omega^1 \geq p_h + 3\delta + 4\epsilon = T(\pi_D) \).

b) \( T(\pi_D) = p_h + 3\delta + 4\epsilon \leq p_h + 2\epsilon + \theta \leq \Omega^2 \).

c) \( \Omega^2 \geq p_h + 2\epsilon + \theta = T(\pi_D) - (3\delta + 2\epsilon - \theta) \).

The condition \( \theta \leq 3\delta + 2\epsilon \) implies that the fraction

\[
T(\pi_D) \leq \frac{p_h + 3\delta + 4\epsilon}{p_h + 2\epsilon + \theta} \Omega^2
\]
is maximized by minimizing $p_h$. Therefore

$$T(\pi_D) \leq \frac{2(m + 1)(\delta + \epsilon)}{(2m - 1)\delta + 2m\epsilon + \theta} \Omega^2 \leq \frac{2(m + 1)(\delta + \epsilon)}{2m(\delta + \epsilon)} \Omega^2 = \frac{m + 1}{m} \Omega^2.$$ 

d) Either $S^2_m$ is optimal (by Lemma 1) or $T(S^2_m) = (m + 2)(\delta + 2(m + 1)\epsilon + m\theta \leq (m + 1)(\delta + \epsilon) \leq p_h + 3\delta + 4\epsilon \leq \Omega^1$. Therefore

$$T(S^2_m) \leq \frac{2(m + 1)(\delta + \epsilon)}{(2m - 1)\delta + 2m\epsilon + \theta} \Omega^2 \leq \frac{2(m + 1)(\delta + \epsilon)}{(2m - 1)\delta + 2m\epsilon} \Omega^2 \leq \frac{2m + 2}{2m - 1} \Omega^2.$$ 

Proof of Lemma 3: If $\theta \leq \delta \leq p_i, \forall i$, then either $S^2_m$ is optimal (by Lemma 1) or $T(S^2_m) = (m + 2)(\delta + 2(m + 1)\epsilon + m\theta \leq 2(m + 1)(\delta + \epsilon) \leq \Omega^1$. That $\theta \leq \delta \leq p_i, \forall i$, implies $T(S^2_m) = \Omega^2$ is proven in Geismar et al. (2006). If $\delta \leq \theta$, and $p_h \leq (2m - 1)\delta + 2(m - 1)\epsilon$, then $T(\pi_D) = 2(m + 1)(\delta + \epsilon) \leq (m + 1)(\delta + 2\epsilon + \theta) \leq \Omega^2$. That $p_i \geq \delta, \forall i$, implies $T(\pi_D) = \Omega^1$ is proven in Dawande et al. (2002).

Proof of Lemma 4: Geismar et al. (2006) show that $S^2_o$ is optimal over all dual-gripper cycles for $p_i \leq (\delta + \theta)/2, \forall i$. Dawande et al. (2002) show that $\pi_U$ is optimal over all single-gripper cycles for $p_i \leq \delta, \forall i$. Recall that $S^2_o$ is a 2-unit cycle. If $\theta \leq \delta$, then

$$T(S^2_o) = \frac{(m + 2)}{2} \delta + 2(m + 1)\epsilon + \frac{(m + 2)}{2} \theta + \sum_{i=1}^{m} p_i \leq (m + 2)\delta + 2(m + 1)\epsilon + \sum_{i=1}^{m} p_i = \Omega^1$$

If $\delta \leq \theta$, then

$$T(\pi_U) = (m + 2)\delta + 2(m + 1)\epsilon + \sum_{i=1}^{m} p_i \leq \frac{(m + 2)}{2} \delta + 2(m + 1)\epsilon + \frac{(m + 2)}{2} \theta + \sum_{i=1}^{m} p_i = \Omega^2.$$ 

Appendix C: Proof of Theorem 2

Proof of Theorem 2:

Step 1: If $p_i \leq \delta, \forall i$, then $\pi_U$ is optimal by Lemma 4.

Step 2: If $\max_{1 \leq i \leq m} p_i + 3\delta + 4\epsilon \geq 2(m + 1)(\delta + \epsilon)$, then $\pi_D$ is optimal by Lemma 2

Step 3: If $p_i \geq \delta, \forall i$, then $\pi_D$ is optimal by Lemma 3.
Step 4: If $|D_δ| \geq \frac{5m-4}{9}$, then, from (3), we have

$$\Omega^1 \geq \left[ m + \frac{5m-4}{9} + 2 \right] \delta + 2(m+1)\epsilon + \sum_{i \in D_δ} p_i$$

$$\geq \left( \frac{14}{9} m + \frac{14}{9} \right) \delta + 2(m+1)\epsilon.$$ 

Thus,

$$T(\pi_D) = 2(m+1)(\delta + \epsilon) \leq \frac{2(m+1)(\delta + \epsilon)}{\left( \frac{14}{9} m + \frac{14}{9} \right) \delta + 2(m+1)\epsilon} \Omega^1 \leq \frac{9}{7} \Omega^1.$$ 

We now show tightness. Suppose $m = 8$, $p_1 = p_3 = p_5 = p_7 = 2\delta$, and $p_2 = p_4 = p_6 = p_8 = \nu < \delta$. An optimal cycle is the basic cycle based on the initial partition: $\pi_1 = (A_0, A_7, A_8, A_5, A_6, A_3, A_4, A_1, A_2)$, $T(\pi_1) = LB_1^1 = 14\delta + 18\epsilon + 4\nu$. Algorithm CCell2 outputs $\pi_D$, and $T(\pi_D) = 18(\delta + \epsilon)$. Therefore, $T(\pi_D)/T(\pi_1) \to 9/7$ as $\epsilon \to 0$ and $\nu \to 0$.

Step 5: The structure of $V_2$ and that $|D_δ| \leq \frac{m+2}{6}$ imply that

$$|V_2| \leq 3|D_δ| \leq \frac{m+2}{2},$$

so

$$\alpha \leq \frac{(m + \frac{m+2}{2} + 2) \delta + 2(m+1)\epsilon + \sum_{i \in V_2} p_i}{(m + |D_δ| + 2)\delta + 2(m+1)\epsilon + \sum_{i \in D_δ} p_i} \Omega^1.$$

$$\leq \frac{\frac{3}{2}(m+2)}{\frac{7}{6}(m+2)} \Omega^1 = \frac{9}{7} \Omega^1.$$ 

We now investigate the value of $\beta_i, i \in V_2$:

1. By construction, for $i \in D_δ$, $\beta_i = p_i + 3\delta + 4\epsilon$. By (4), if $T(\tilde{\pi}_B) = p_i + 3\delta + 4\epsilon$ for some $i$, then $\tilde{\pi}_B$ is optimal.

2. For $i \in V_2 \setminus D_δ$, $\beta_i = p_i + 3\delta + 4\epsilon + \sum_{j \in X_i \cup Y_i} (p_j + \delta + 2\epsilon)$. Since $i \in D_δ^c$, $X_i \subset D_δ^c$, and $Y_i \subset D_δ^c$, we know that $p_i + \sum_{j \in X_i \cup Y_i} p_j \leq \sum_{j \in D_δ^c} p_j$. Cycle $\tilde{\pi}_B$ was designed so that $|V_2| \geq 2$. This implies that $|X_i \cup Y_i| \leq m - 2$. Hence,

$$\beta_i \leq \sum_{j \in D_δ^c} p_j + (m - 2)(\delta + 2\epsilon)$$

$$= \sum_{j \in D_δ^c} p_j + (m + 1)\delta + 2m\epsilon, i \in V_2 \setminus D_δ.$$
This value is strictly less than $LB_1^1$. Hence, $\beta_i, i \in V_2 \setminus D_\delta$, will not dominate the cycle time expression.

**Step 6:** We show that $\alpha/LB_1^1 \leq \frac{9}{7}$ by first proving that

$$|V_2 \setminus D_\delta| \leq \frac{2}{7}(m + |D_\delta| + 2), \quad (6)$$

i.e., by establishing a bound on the number of elements of $D_\delta^c$ that the algorithm places into $V_2$. The largest value for $|V_2 \setminus D_\delta|$ is obtained by maximizing the number of elements $i \in D_\delta$ for which $|Y_i| > \frac{9}{56}(m + |D_\delta| + 2) + 1$. For each such $i$, two elements of $Y_i$ (the first and the last) will be added to $V_2 \setminus D_\delta$. For example, consider the cell in Figure 5: $m = 12, D_\delta = \{1, 6, 11, 12\}$, so $|D_\delta| = 4$ and $\frac{9}{56}(m + |D_\delta| + 2) + 1 = 3.89$. $|Y_1| = |Y_6| = 4$, so indices 2, 5, 7, and 10 will be added to $V_2$. In general, such a maximal $|V_2 \setminus D_\delta|$ is formed in an $m$-machine cell if

$$D_\delta = \left\{1, 2 + \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \right.$$  

$$3 + 2 \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \right.$$  

$$\vdots$$  

$$z + (z - 1) \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \right.$$  

$$z + (z - 1) \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor + 1, \ldots, m \right\}.$$

Note that this uniquely defines the integer $z$ as

$$z = \left\lfloor \frac{m}{\left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 3 \right\rfloor} \right\rfloor.$$
and the size of $D_\delta$ as

$$|D_\delta| = m - (z - 1) \left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 2 \right\rfloor.$$

($z = 3$ and $|D_\delta| = 4$ for the example in Figure 5.) In such a cell, $|V_2 \setminus D_\delta| = 2(z - 1)$, where $|Y_{i,j}| = \left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 2 \right\rfloor$ for $i_j \in D_\delta$, $j = 1, \ldots, z - 1$, and $|Y_{i,j}| = 0$, for $i_j \in D_\delta$, $j = z, \ldots, |D_\delta|$, and $|X_{i,j}| = 0$. Thus, there are $z - 1$ intervals in which the first machine is an element of $D_\delta$ and the next $\left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 2 \right\rfloor$ machines are elements of $D_\delta^c$. (In Figure 5, $M_1$ and $M_6$ are elements of $D_\delta$ that are each followed by four elements of $D_\delta^c$.) The final $|D_\delta| - z + 1$ machines ($M_{11}, M_{12}$) are elements of $D_\delta$. Hence, this requires that there are at least $(z - 1) \left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 3 \right\rfloor + |D_\delta| - z + 1 = (z - 1) \left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 2 \right\rfloor + |D_\delta|$ machines in the cell. To prove (6) we must show that our algorithm cannot add another machine to $V_2 \setminus D_\delta$, if $2(z - 1) = \frac{2}{7} (m + |D_\delta| + 2)$. For the algorithm to add another machine to $V_2 \setminus D_\delta$, there must be an additional $\left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 1 \right\rfloor$ elements of $D_\delta^c$ between any pair of the last $|D_\delta| - z + 1$ elements of $D_\delta$. We claim that such a configuration is infeasible. If it were feasible, then

$$(z - 1) \left\lfloor \frac{9}{56} (m + |D_\delta| + 2) + 2 \right\rfloor + |D_\delta| + 2 \leq m$$

$$\Rightarrow (z - 1) \left( \frac{9}{56} (m + |D_\delta| + 2) + 1 \right) + |D_\delta| + \frac{9}{56} (m + |D_\delta| + 2) \leq m. \quad (7)$$

Because $|D_\delta| > \frac{m+2}{6}$, we have $m + |D_\delta| + 2 > \frac{7}{6}(m + 2)$ and

$$z - 1 = \frac{1}{7}(m + |D_\delta| + 2) > \frac{m+2}{6}. \quad \text{Thus, (7) must satisfy}$$

$$\frac{m+2}{6} \left( \frac{9}{56} \cdot \frac{7}{6} (m + 2) + 1 \right) + \frac{m+2}{6} + \frac{9}{56} \cdot \frac{7}{6} (m + 2) \leq m$$

$$(m+2)(3m+22) + 34(m+2) \leq 96m$$

$$3m^2 - 34m + 112 \leq 0,$$

which is a contradiction. Therefore, $|V_2 \setminus D_\delta| \leq \frac{4}{7} (m + |D_\delta| + 2)$. 


Thus,
\[
\frac{\alpha}{LB_1^i} \leq \frac{(m + |D_\delta| + \frac{3}{2}(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon + \sum_{j \in V_1} p_j)}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon + \sum_{j \in D_\delta^i} p_j}
\]
\[
\leq \frac{\frac{9}{7}(m + |D_\delta| + 2)}{m + |D_\delta| + 2} = \frac{9}{7}.
\]

We now show that \(\beta_i \leq \frac{9}{7} \max\{LB_1^1, LB_1^2\}\), for all \(i \in V_2\). First, either \(\beta_i \leq \frac{9}{7} LB_1^2\) or \(\beta_i LB_1^2 = p_i + 3\delta + 4\epsilon + \sum_{j \in X_i \cup Y_i} (p_j + \delta + 2\epsilon) > \frac{9}{7}\)
\[
\Leftrightarrow 7 \sum_{j \in X_i \cup Y_i} (p_j + \delta + 2\epsilon) > 2(p_i + 3\delta + 4\epsilon).
\]

Therefore,
\[
\frac{\beta_i}{LB_1^1} \leq \frac{\frac{9}{7} \sum_{j \in X_i \cup Y_i} (p_j + \delta + 2\epsilon)}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon + \sum_{i \in D_\delta^i} p_i}
\]
\[
= \frac{\sum_{j \in X_i \cup Y_i} \left(\frac{7}{2} p_j + \frac{9}{2}(\delta + 2\epsilon)\right) + \sum_{j \in X_i \cup Y_i} p_j}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon + \sum_{i \in D_\delta^i} p_i}
\]
\[
\leq \frac{\sum_{j \in X_i \cup Y_i} \left(\frac{7}{2} p_j + \frac{9}{2}(\delta + 2\epsilon)\right)}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon}, \quad [X_i \cup Y_i \subset D_\delta^i]
\]
\[
< \frac{|X_i \cup Y_i|(8\delta + 9\epsilon)}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon}, \quad [p_j < \delta, \ j \in X_i \cup Y_i]
\]

Because \(|X_i \cup Y_i| \leq \frac{9}{56}(m + |D_\delta| + 2)\), for \(i \in D_\delta\), and \(|D_\delta| < \frac{5m-4}{9}\) (by Step 4),
\[
\frac{\beta_i}{LB_1^1} \leq \frac{(m + |D_\delta| + 2)(8 \cdot \frac{9}{56} \delta + 9 \cdot \frac{9}{56} \epsilon)}{(m + |D_\delta| + 2)\delta + 2(m + 1)\epsilon} \leq \frac{9}{7}.
\]

Appendix D: Proof of Lemma 5

**Proof of Lemma 5:** We find the worst case bound and show that it is less than or equal to 3/2. The coefficient of \(\theta\) in \(T(S_a)\) (equation (5)) cannot be greater than its coefficient in \(2LB_1^2\), so the largest value for \(T(S_a)/2LB_1^2\) occurs when \(\theta = 0\). Since \(D_2^i = U_5 \cup U_6 \cup U_7 \cup U_9\) in cycle...
\[ T(S_a) \leq \frac{(2m + 4 - u_0 - u_1 - u_9)\delta + 4(m + 1)\epsilon + \sum_{i \in U_7 \cup U_9} p_i + 2 \sum_{i \in D_2} p_i}{[2(m + 1) - |D_2|]\delta + 4(m + 1)\epsilon + 2 \sum_{i \in D_2} p_i} \]

First consider the case in which \( U_9 = \emptyset \). The following cell has the largest \(|D_2^e|\) for which algorithm DGR-Cell assigns no machine to \( U_9 \): \( m = 3k \), \( p_{3j-2} < (\delta + \theta)/2 \), \( p_{3j} < (\delta + \theta)/2 \), \( p_{3j-1} \geq (\delta + \theta)/2 \) (but the \( p_{3j-1} \) are not large enough to cause positive partial waiting), \( j = 1, \ldots, k \), so \(|D_2| = k\) and \(|D_2^e| = 2k\). Thus, this cell has \( k-1 \) runs with \( s = 2 \), plus \( \{1, m\} \subset D_2^e \), so the algorithm makes the assignments 58668668····86687. Thus, bounding inequality (8) becomes

\[ \frac{T(S_a)}{2LB_1^2} \leq \frac{2m + 4 - u_0 - u_1 - u_9}{2(m + 1) - |D_2|} \leq \frac{6k + 2}{4k + 2} < \frac{3}{2}. \]

Now suppose \( U_9 \neq \emptyset \). If a run with length \( s \geq 2 \) begins at \( r_1 = 1 \) or ends at \( r_G + s_G - 1 = m \), then \( s - 1 \) elements (i.e., at minimum one-half of the elements) of that run are placed into \( U_9 \).

For any other run with length \( s \geq 3 \), \( s - 2 \) elements (i.e., at minimum one-third of the elements) of that run are placed into \( U_9 \). Therefore, because we want to maximize \(|D_2^e| - u_9\), we consider only those runs with \( s = 3 \) and \( r_1 \neq 1 \), \( r_G + s_G - 1 \neq m \). In each of these \( G \) runs, for \( g = 1, \ldots, G \), we have \( r_g \in U_7 \), \( r_{g+1} \in U_9 \), \( r_{g+2} \in U_5 \); \( r_{g+3} \in U_8 \) to separate the runs. Hence, the cell that maximizes \(|D_2^e| - u_9\) and therefore has the largest value for (8) causes algorithm DGR-Cell to make case assignments so that the cycle begins with one trio assigned 586 and \( \ell - 1 \) trios assigned 686, has a machine in Case 8 (to separate the runs), then has \( G \) quartets assigned 7958, and concludes with \( j - \ell - 1 \) trios assigned 686 and one trio assigned 687. Therefore, \( m = 4G + 3j + 1 \), \( |D_2^e| = 3G + 2j \), \( u_9 = G \), so

\[ \frac{T(S_a)}{2LB_1^2} \leq \frac{2(4G + 3j + 1) + 4 - G}{2(4G + 3j + 2) - 3G - 2j} = \frac{7G + 6j + 6}{5G + 4j + 4} < \frac{3}{2}. \]

**Appendix E: Proof of Lemma 7**

**Proof of Lemma 7:** We use the fact that if \( q \in U_8 \), then \( M_q \) is reloaded immediately after
it is unloaded. We can write the waiting time expressions for any \( q \in U_8 \) as follows:

\[
w^1_q = \max \left\{ 0, \hat{w}^1_q - \sum_{j=q+1}^m w^1_j - \sum_{j=1}^{q-1} w^1_j \right\}
\]

\[
w^2_q = \max \left\{ 0, \hat{w}^2_q - \sum_{j=q+1}^m w^2_j - \sum_{j=1}^{q-1} w^2_j \right\},
\]

where \( \hat{w}^1_q \) and \( \hat{w}^2_q \) are expressions of the form

\[
\hat{w}^1_q = p_q - a\delta - b\epsilon - c\theta - \sum_{j=q+1}^m p_j - \sum_{j=1}^m p_j - \sum_{j=q+1}^m p_j - \sum_{j=1}^{q-1} p_j - \sum_{j=q+1}^m p_j,
\]

\[
\hat{w}^2_q = p_q - d\delta - e\epsilon - f\theta - \sum_{j=1}^q p_j - \sum_{j=1}^m p_j - \sum_{j=1}^{q-1} p_j - \sum_{j=1}^{q-1} p_j - \sum_{j=1}^{q-1} p_j,
\]

and \( a, b, c, d, e, f \geq 0 \) are constants. From (9) and (10) we get

\[
\sum_{j=q+1}^m w^2_j + \sum_{j=1}^q w^1_j = \max \left\{ \sum_{j=q+1}^m w^2_j + \sum_{j=1}^q w^1_j, \hat{w}^1_q \right\}
\]

\[
\sum_{j=q+1}^m w^1_j + \sum_{j=1}^q w^2_j = \max \left\{ \sum_{j=q+1}^m w^1_j + \sum_{j=1}^q w^2_j, \hat{w}^2_q \right\}.
\]

Therefore, if \( w^i_q > 0 \), then \( \hat{w}^i_q \) is the robot’s total partial waiting time while \( M_i^q \) is processing, \( i = 1, 2 \). These two equations also imply the following system of 2\( u_8 \) inequalities:

\[
\sum_{j=q+1}^m w^2_j + \sum_{j=1}^q w^1_j \geq \max \left\{ 0, \hat{w}^1_q \right\}, \quad q \in U_8,
\]

\[
\sum_{j=q+1}^m w^1_j + \sum_{j=1}^q w^2_j \geq \max \left\{ 0, \hat{w}^2_q \right\}, \quad q \in U_8.
\]
Let $q' = \arg\max_{q \in U_S} \left\{ \max\{0, \hat{w}_q^1\} + \max\{0, \hat{w}_q^2\} \right\}$. We show that $W = w_q^1 + w_q^2$, where $w_q^1 = \max\{0, \hat{w}_q^1\}$, $w_q^2 = \max\{0, \hat{w}_q^2\}$, and $w_q^1 = w_q^2 = 0$ for $q \neq q'$, is a minimal solution to the system of inequalities (11) and (12). To prove this, we show that (a) $W \geq w_q^1 + w_q^2$, and that (b) $W = w_q^1 + w_q^2$ is a feasible solution to the system of inequalities (11) and (12):

(a) That $W \geq w_q^1 + w_q^2$ follows from summing inequalities (11) and (12) for $q = q'$:

$$W = \sum_{j=q'+1}^{m} w_j^2 + \sum_{j=1}^{q'} w_j^1 + \sum_{j=1}^{m} w_j^1 + \sum_{j=q'+1}^{q} w_j^2 \geq \max\{0, \hat{w}_q^1\} + \max\{0, \hat{w}_q^2\}. $$

(b) Suppose $q > q'$. For (11) and (12) to be satisfied, we must have $w_q^1 \geq \hat{w}_q^1$ and $w_q^2 \geq \hat{w}_q^2$, which follow from (9) and (10):

$$0 = w_q^1 = \max\{0, \hat{w}_q^1 - w_q^1\} \Rightarrow w_q^1 \geq \hat{w}_q^1,$$

$$0 = w_q^2 = \max\{0, \hat{w}_q^2 - w_q^2\} \Rightarrow w_q^2 \geq \hat{w}_q^2.$$

If $q < q'$, then we must have $w_q^2 \geq \hat{w}_q^1$ and $w_q^1 \geq \hat{w}_q^2$, which follow similarly from (9) and (10).

By Lemma 6, if $w_q^1 > 0$ and $w_q^2 > 0$, then $S_a$ is optimal. Otherwise, $W = w_q^1 > 0$ and $w_q^3-i = 0$, $i \in \{1, 2\}$. 

**Appendix F: Proof of Theorem 3**

**Proof of Theorem 3:** If $\max_{1 \leq i \leq m} p_i \geq (m + 2)\delta + 2m\epsilon + (m - 1)\theta$, then cycle $S_m^2$ is optimal by Lemma 1. If $p_i \geq \delta, \forall i$, then cycle $S_m^2$ is optimal by Lemma 3. If $p_i \leq (\delta + \theta)/2, \forall i$, then cycle $S_o^2$ is optimal by Lemma 4. If $W = 0$ in cycle $S_a$, then $S_a$ provides a 3/2-approximation to the optimal per unit cycle time by Lemma 5.

We now analyze the case in which $W > 0$ in cycle $S_a$. The proof of Lemma 7 shows that we need only to consider $W = w_q^1$, $q \in U_S$, where either $i = 1$ or $i = 2$. First suppose that $w_q^1 > 0$. In this case, we can find $T(S_a)$ by adding $p_q$ and the times for robot actions and full waiting that occur between the start of the unloading of $M^i_q$ and the completion of the loading of this usage.
We first compute the robot movement times in this expression for $T(S_a)$. After unloading $M_q^1$, rotating its grippers, and loading $M_q^2$, the robot travels to each machine $M_j$, $j > q$, $j \in U_5 \cup U_6 \cup U_7 \cup U_8$, then to $O$ (if $u_1 = 0$), to $I$, and to each machine $M_1, \ldots, M_q$, before unloading $M_q^2$, rotating its grippers, and reloading $M_q^1$. Hence, the total movement time in $S_a$ is $(q + 2 - u_1 + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}|)\delta$.

Now consider the load/unload, gripper rotation, and full waiting times. For clarity, we explicitly calculate only the load/unload times within the text. After unloading $M_q^1$ but before visiting $I$, the robot unloads each $M_j$ for which $j \in U_5$ and $j > q$, and it loads each $M_j$ for which $j \in U_7$ and $j > q$ (this requires time $|\{j : j > q, j \in U_5 \cup U_7\}|\epsilon$). During this same segment of the cycle, the robot loads, waits, and unloads (respectively, unloads, rotates grippers, and loads) at each $M_j$, $j \in U_6$ (resp., $j \in U_8$), $j \geq q$ (time $2|\{j : j \geq q, U_5 \cup U_7\}|\epsilon$). Before visiting $I$, the robot may load $O$ (time $(1 - u_1)\epsilon$). After visiting $I$ (time $(1 + u_0)\epsilon$) but before loading $M_q^1$, the robot loads, waits, unloads, rotates grippers, and reloads (respectively unloads, rotates grippers, reloads, waits, and unloads) at each $M_i$, $i \in U_5$ (resp., $i \in U_7$), $i < q$ (time $3|\{i : i < q, i \in U_5 \cup U_7\}|\epsilon$). At each $M_i$, $i \in U_6$ (resp., $i \in U_8$), $i \leq q$, the robot loads, waits, and unloads (resp., unloads, rotates grippers, and loads) (time $2|\{i : i \leq q, U_5 \cup U_7\}|\epsilon$).

For each machine $M_i$, $i \in U_9$, $i < q$, the robot loads, waits, unloads, rotates grippers, reloads, waits, and unloads (time $4|\{i : i < q, i \in U_9\}|\epsilon$). This analysis generates the following cycle time expression if $W = w_q^1$:

$$T(S_a) = p_q + (q + 2 - u_1 + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}|)\delta$$

$$+ [4|\{i : i < q, i \in U_6\}| + 3|\{i : i < q, i \in U_5 \cup U_7\}| + 2(|U_6 \cup U_8| + 1)$$

$$+ |\{j : j > q, j \in U_5 \cup U_7\}| + 2 + u_0 - u_1]\epsilon$$

$$+ (|U_8| + |\{i : i < q, i \in U_5 \cup U_7 \cup U_9\}| + u_0 + 1)\theta$$

$$+ 2 \sum_{i=1}^{q-1} p_i + \sum_{i=1}^{q-1} p_i + \sum_{i \in U_6} p_i.$$

Note that $q + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}| \leq m$, $u_5 + u_7 + u_8 + u_9 \leq m$, and $0 \leq u_0, u_1 \leq 1$. 

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Thus,

\[ T(S_a) \leq p_q + (m + 2)\delta + 4(m + 1)\epsilon + (m + 2)\theta + 2\sum_{i \in gU_8} p_i. \]  

(13)

We further bind \( T(S_a) \) by binding \( p_q \) by using \( LB_2^2 \). If \( T(S_a) / 2LB_2^2 \leq 3/2 \), then the theorem is proven. Otherwise, \( T(S_a) > 3p_q + 3\theta + 6\epsilon \), so, by (13), we have

\[ 3p_q + 3\theta + 6\epsilon < p_q + (m + 2)\delta + 4(m + 1)\epsilon + (m + 2)\theta + 2\sum_{i \in gU_8} p_i. \]

\[ p_q < \frac{m + 2}{2}\delta + (2m - 1)\epsilon + \frac{m - 1}{2}\theta + \sum_{i \in gU_8} p_i. \]

\[ \Rightarrow T(S_a) < \frac{3}{2}(m + 2)\delta + (6m + 3)\epsilon + \frac{3}{2}(m + 1)\theta + 3\sum_{i \in gU_8} p_i. \]

Recall that

\[ 2LB_1^2 = (2m + 2 - |D_2^5|)(\delta + \theta) + 4(m + 1)\epsilon + 2\sum_{i \in D_2^5} p_i \]

\[ = (2m + 2 - u_5 - u_6 - u_7 - u_9)(\delta + \theta) + 4(m + 1)\epsilon + 2\sum_{i \in gU_8} p_i. \]

Since \( u_5 + u_6 + u_7 + u_9 \leq m - 1 \), it follows that

\[ 2LB_1^2 \geq (m + 3)(\delta + \theta) + 4(m + 1)\epsilon + 2\sum_{i \in gU_8} p_i. \]

Therefore,

\[ \frac{T(S_a)}{LB_1^2} < \frac{3}{2}(m + 2)\delta + (6m + 3)\epsilon + \frac{3}{2}(m + 1)\theta + 3\sum_{i \in gU_8} p_i \]

\[ \Rightarrow \frac{T(S_a)}{LB_1^2} < \frac{3}{2}. \]

The proof for \( w_2^2 > 0 \) is similar.

To demonstrate asymptotic tightness, let \( p_i = \nu < (\delta + \theta)/2 \) for \( i = 1, \ldots, m - 1 \), and \( p_m = \frac{m+1.9}{2}\delta + \frac{m-1}{2}\theta \). It follows that

\[ T(S_a) = p_m + (m + 2)\delta + (4m + 2)\epsilon + (m + 2)\theta + (2m - 3)\nu \]

\[ = \left( \frac{3}{2}m + 2.95 \right)\delta + (4m + 2)\epsilon + \frac{3}{2}(m - 1)\theta + (2m - 3)\nu, \]

\[ 2LB_1^2 = (m + 3)(\delta + \theta) + 4(m + 1)\epsilon + 2(m - 1)\nu \]

\[ 2LB_2^2 = 2p_m + 4\epsilon + 2\theta = (m + 1.9)\delta + 4\epsilon + (m + 1)\theta. \]
Therefore,

\[
\frac{T(S_a)}{2LB_1^2} = \frac{\left(\frac{3}{2}m + 2.95\right) \delta + (4m + 2)\epsilon + \frac{3}{2}(m - 1)\theta + (2m - 3)\nu}{(m + 3)(\delta + \theta) + 4(m + 1)\epsilon + 2(m - 1)\nu} \rightarrow \frac{3}{2}
\]

\[
\frac{T(S_a)}{2LB_2^2} = \frac{\left(\frac{3}{2}m + 2.95\right) \delta + (4m + 2)\epsilon + \frac{3}{2}(m - 1)\theta + (2m - 3)\nu}{(m + 1.9)\delta + 4\epsilon + (m + 1)\theta} \rightarrow \frac{3}{2}
\]

as \(m \rightarrow \infty\), \(\epsilon \rightarrow 0\), and \(\nu \rightarrow 0\).