9 Appendix: Proofs

Proof of Theorem 1. i. According to Lemma 1 in Zhao and Atkins (2007), we need to show \( \pi_i(p_i) = (w_i - \beta_i)(k - mp_i) + (p_i - \beta_i)E[\min\{D_i^g, k - mp_i\}] \) to be quasiconcave in \( p_i \).

\[
\frac{d\pi_i(p_i)}{dp_i} = \frac{d\pi_i^d}{dp_i} + (w_i - \beta_i)m + E[\min\{D_i^g, k - mp_i\}] - m(p_i - \beta_i)\Pr(D_i^g > k - mp_i)
\]

\[
\frac{d^2\pi_i(p_i)}{dp_i^2} = \frac{d^2\pi_i^d}{dp_i^2} + \Pr(D_i^g > k - mp_i)[-2m - m^2(p_i - \beta_i)r_{D_i^g}(k - mp_i)]
\]

If \( m \geq 0 \), then \( \pi_i(p_i) \) is strictly concave in \( p_i \), done.

If \( m < 0 \), then let \( n = -m > 0 \). According to (A), \( \frac{d^2\pi_i^d}{dp_i^2} < 0 \) and is decreasing in \( p_i \). If \([2 - n(p_i - \beta_i)r_{D_i^g}(k + np_i)] < 0 \), then \( \pi_i(p_i) \) is strictly concave in \( p_i \), done. Otherwise, by (B), \([2 - n(p_i - \beta_i)r_{D_i^g}(k + np_i)] \) decreases as \( p_i \) increases from \( w_i \) to \( p_i^{\text{max}} \). Hence \( \frac{d^2\pi_i(p_i)}{dp_i^2} \) either changes sign at most once from positive to negative or is always negative. Thus, whenever \( \frac{d\pi_i(p_i)}{dp_i} \) turns negative, it remains negative, and \( \pi_i(p_i) \) is quasiconcave in \( p_i \).

So, function (1) is quasiconcave in \((p_i, y_i)\) and a pure-strategy Nash equilibrium exists.

ii. We first show that maxima of function (1) are interior, then that equations (2)-(3) have a unique solution.

Note that \( \lim_{p_i \to p_i^{\text{max}}} \frac{d\pi_i}{dp_i} < 0 \), \( \lim_{y_i \to y_i^{\text{max}}} \frac{d\pi_i}{dy_i} = -(w_i - \beta_i) < 0 \), \( \lim_{p_i \to w_i} \frac{d\pi_i}{dp_i} > 0 \), and \( \lim_{y_i \to 0} \frac{d\pi_i}{dy_i} = p_i - w_i > 0 \). So boundary solutions are not optimal. Next we show that a unique maximizer solves (2)-(3), satisfying \( Q(p_i) = \frac{\partial^2\pi_i^d}{\partial p_i^2} + \Pr(D_i^g > y_i)/[(p_i - \beta_i)r_{D_i^g}(y_i)] < 0 \).

\[4\]Proved in Zhao and Atkins (2007), a bivariate function \( g(x_1, x_2) \) is jointly quasiconcave in two variables iff every “vertical slice” of the function is quasiconcave, or more formally, iff \( g(x_1, x_2) \) is quasiconcave given \( mx_1 + x_2 = k \) for any real values \( m \) and \( k \).
Uniquely solve $y_i(p_i)$ from (3) and substitute into (2), resulting in

$$\frac{\partial \pi_i^d}{\partial p_i} + E[\min\{D_i^s, y_i(p_i)\}] = 0$$

(A1)

Define $J(p_i) = \frac{d}{dp_i} \pi_i^d + E[\min\{D_i^s, y_i(p_i)\}]$, where $J(w_i) > 0$ and $J(p_i^{\text{max}}) < 0$, and $dJ(p_i)/dp_i = Q(p_i)$. Note that the last term of $Q(p_i)$ decreases with $p_i$ and approaches zero by (B). Also, if (A) holds, then $d^2J(p_i)/dp_i^2 < 0$, so $Q(p_i)$ decreases with $p_i$ and approaches $\partial^2 \pi_i^d / \partial p_i^2$ as $p_i$ goes to $p_i^{\text{max}}$. Thus $J(p_i)$ is strictly concave, starts positive, and finally strictly decreases to negative. So there is a unique solution for equation (A1), at $Q(p_i) < 0$.

**Proof of Proposition 1.** Redefine retailer $j$’s strategy space as $\tilde{y}_j = -y_j$ and $\tilde{p}_j = -p_j$. It can be shown that $\partial^2 \pi_i / \partial p_i \partial y_i \geq 0$, $\partial^2 \pi_i / \partial p_i \partial \tilde{p}_j = 0$, $\partial^2 \pi_i / \partial p_i \partial \tilde{y}_j = 0$, $\partial^2 \pi_i / \partial y_i \partial y_i \geq 0$, and $\partial^2 \pi_i / \partial y_j \partial \tilde{p}_j = 0$. So $\pi_i$ is supermodular in $(p_i, y_i)$ and has increasing difference in $(p_i, \tilde{p}_j)$, $(p_i, \tilde{y}_j)$, $(y_i, \tilde{p}_j)$ and $(y_i, \tilde{y}_j)$. Similarly we show the supermodularity and increasing difference of $\pi_j$. According to Milgrom and Roberts (1990), Theorem 4, the game is supermodular, and a pure Nash equilibrium exists (Topkis 1998, Theorem 4.2.1).

**Proof of Theorem 2.** A sufficient condition (Contraction Mapping Theorem 3.4, Friedman 1990) requires $|\frac{\partial^2 \pi_i}{\partial p_i^2}| > \sum_{j \neq i} \left( |\frac{\partial^2 \pi_i}{\partial p_i \partial y_j} + |\frac{\partial^2 \pi_i}{\partial p_i \partial \tilde{p}_j} + |\frac{\partial^2 \pi_i}{\partial p_i \partial \tilde{y}_j} + \sum_{j \neq i} |\frac{\partial^2 \pi_i}{\partial y_j \partial \tilde{p}_j} \right)$ for uniqueness, which are

$$-\frac{\partial^2 \pi_i^d}{\partial p_i^2} > \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial D_j} + \Pr(D_i^s > y_i) + \sum_{j \neq i} \gamma_j \Pr(D_i^s < y_i, \epsilon_j > y_j)$$

$$1 > \frac{1}{[p_i - \beta_i] r_{D_i}} + \sum_{j \neq i} \gamma_j f_{D_i^s|\epsilon_j > y_j}(y_i) \Pr(\epsilon_j > y_j) / f_{D_i^s}(y_i)$$

The required results is obtained by

(i) $1/[\beta_i - \beta_i] r_{D_i} \leq 1/[w_i - \beta_i] r_{D_i}$,
Thus a unique symmetric equilibrium exists. 

Here $D_i$ have to be negative and stay negative. Then there is a unique symmetric equilibrium. That is, the solution from (2) and (3) under symmetry, $s$ is unique symmetric equilibrium for the game (Cachon and Netessine 2004). Now we show that given $D_i = \lambda_i$ decreases in $\gamma_i$. Define $l_i = \frac{\gamma_i \Pr(D_i > y_i) + \sum_{j \neq i} \gamma_{ji} \Pr(D_i > y_i, \epsilon_j > y_j)}{\gamma_i \Pr(D_i > y_i, \epsilon_j > y_j)} \leq \max\{1, \sum_{j \neq i} \gamma_{ji}\}$. 

**Proof of Proposition 2.** An immediate result from Theorem 1 is that there exists a symmetric equilibrium for the game (Cachon and Netessine 2004). Now we show that given $p_i = p_{-i} = p$ and $y_i = y_{-i} = y$ and a symmetric demand and cost function, there exists a unique symmetric equilibrium. That is, the solution from (2) and (3) under symmetry,

\[-(w - \beta) + (p - \beta) \Pr(D_i \geq y) = 0 \quad \text{(A2)}\]

\[\partial^2 \pi_i^d / \partial p_i^2 + E[\min\{D_i, y\}] = 0, \quad \text{(A3)}\]

is unique. Define $J(p) = \partial^2 \pi_i^d / \partial p_i^2 + E[\min\{D_i, y(p)\}]$, where $y(p)$ is the unique solution of equation (A2). Now $J(w) > 0$, and $J(p_{\text{max}}) < 0$. Also $dJ(p)/dp = \partial^2 \pi_i^d / \partial p_i^2 + \sum_{j \neq i} \partial^2 \pi_i^d / \partial p_i \partial p_j + A(y)p'(p)$, where $A(y) = \partial E[\min\{D_i, y\}] / \partial y$ and $y'(p) = dy(p)/dp$.

First, we show that $A(y) > 0$ and decreases in $y$. Then we show that $y'(p) > 0$ and decreases in $y$. Then $dJ(p)/dp$ can be either always negative, or start positive but decrease to negative and stay negative. Then there is a unique $p$ that solves $J(p) = 0$, at $dJ(p)/dp < 0$. Thus a unique symmetric equilibrium exists.

Using a methodology introduced by Netessine and Rudi (2003) for differentiation, we have $A(y) = \partial E[\min\{D_i, y\}] / \partial y = \Pr(D_i \geq y_i) - (N - 1) \gamma \Pr(D_i < y_i, \epsilon_j > y_j)$

$\geq \Pr(D_i > y_i) - \Pr(D_i < y_i, \epsilon_j > y_j) \geq \Pr(D_i > y_i) - \Pr(\epsilon_i > y_i) \geq 0$. 

Also note that $E[\min\{D_i, y\}] = E[\min\{\epsilon_i, y\}] + E \min\{(y - \epsilon_i)^+, (N - 1)\gamma(\epsilon_j - y)^+\}$. So $A(y) = \partial (E[\min\{\epsilon_i, y\}] + E \min\{(y - \epsilon_i)^+, (N - 1)\gamma(\epsilon_j - y)^+\}) / \partial y$

$= \Pr(\epsilon_i > y) + \Pr(y - (N - 1)\gamma(\epsilon_j - y) < \epsilon_i < y) - (N - 1)\gamma \Pr(y < \epsilon_j < y + (y - \epsilon_i)/(N -}$

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1)\(γ\), which decreases with \(y\).

From (4.2), \(y'(p) = \Pr(D^*_i > y)/(p - \beta)(\partial \Pr(D^*_i > y)/\partial y) = 1/(p - \beta)r_{D^*_i}(y)\), which decreases in \(y\) under the IFR assumption for \(D^*_i\) and the fact that \(D^*_i\) stochastically decreases with \(y\).

**Proof of Theorem 3.** Given \((p^c_i, y^c_i)\), the unique best response of retailer \(i\) will be \((p^*_i, y^*_i)\) if functions (2)-(3) are equivalent to (4)-(5). Thus, \(w^*_i = c_i - \sum_{j \neq i}(p^c_j - c_j)L_j^i(p^r)/L^i(p^r)\) and \(\beta^*_i = [w^*_i - p^c_i \Pr(D^*_i > y^c_i)]/\Pr(D^*_i < y^c_i) = p^c_i - (p^c_i - w^*_i)/\Pr(D^*_i < y^c_i).\) This approach has been justified by Winter (1993), Cachon (1999), and Tsay and Agrawal (2000). It can be shown that \(c_i < w^*_i < p^c_i\), and \(\beta^*_i = p^c_i + (L_i(p^r) + E[\min(D^*_i, y^c_i)])/[L^i(p^r) \Pr(D^*_i < y^c_i)] < w^*_i\).

Next, we prove that \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\) is a Pareto-dominant equilibrium for the whole game.

Assume there is another equilibrium \((\overrightarrow{p^c_i}, \overrightarrow{y^c_i})\) that Pareto-dominates \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\). Then at \((\overrightarrow{p^c_i}, \overrightarrow{y^c_i})\), at least one player gets better off without making any other player worse off than at \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\). But this is not possible since at \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\), the total supply chain’s profit is no less than that at \((\overrightarrow{p^c_i}, \overrightarrow{y^c_i})\). If one player is better off at \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\), there must be at least one player getting worse off at \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\). So \((\overrightarrow{p^*_i}, \overrightarrow{y^*_i})\) is a Pareto-dominant equilibrium.

Assume that the optimum for the system is the unique. If the payoffs are transferrable among players, then similar reasoning shows that it is the unique Pareto-dominant equilibrium.

**Proof of Proposition 3.** With price competition only, \(\beta^*_i = p^c_i - (p^c_i - w^*_i)/\Pr(D^*_i < y^c_i) = (-c_i + w^*_i)/\Pr(D^*_i < y^c_i) > 0.\) With inventory competition only, \(w^*_i = c_i\) and \(\beta^*_i = -\sum_{j \neq i}p^c_j \gamma_j \Pr(D^*_j < y^c_j, \varepsilon_i > y^c_i)/\Pr(D^*_i < y^c_i) < 0.\)

**Proof of Proposition 4.**

(i) To simplify the presentation, let \(H = \text{def} E[\min(D^*_i, y_i)].\) Then \(\partial H/\partial y_i = \Pr(D^*_i > y_i)\)
and $\partial H/\partial y_j = -\gamma \Pr(D_i^s < y_i, \epsilon_j > y_j)$. We first show that at a symmetric equilibrium (solution of (A2) and (A3)), we have

$$\frac{\partial^2 \pi_i^d}{\partial p_i} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} - \left( \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} \right) \frac{\partial H/\partial y_i}{(p-\beta)(\partial^2 H/\partial y_i^2 + \sum_{j \neq i} \partial^2 H/\partial y_i \partial y_j)} < 0. $$

Following Theorem 1, the symmetric equilibrium price is solved by equation (A1). That is, $J(p) = \partial \pi_i^d/\partial p_i + E[\min\{D_i^s, y(p)\}] = 0$, where $y(p)$ is the solution to equation (3) after setting $y_i = y$ for all $i$. As in part ii of the proof of Theorem 1, the solution $p$ to $J(p) = 0$ must occur when $dJ(p)/dp < 0$. Note that

$$\frac{dJ(p)}{dp} = \frac{\partial^2 \pi_i^d}{\partial p_i} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} + \left( \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} \right) \frac{dy_p}{dp}$$

where $\frac{dy_p}{dp} = -\frac{\partial H/\partial y_i}{(p-\beta)(\partial^2 H/\partial y_i^2 + \sum_{j \neq i} \partial^2 H/\partial y_i \partial y_j)}$ is derived from equation (3). Hence this intermediate result.

The main result can now be derived. Differentiating (2) and (3) with respect to $\beta$, we have

$$\left( \frac{\partial^2 \pi_i^d}{\partial p_i} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} \right) \frac{dp^*_\beta}{dp} + \left( \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} \right) \frac{dy_p}{d\beta} = 0 \text{ and}$$

$$\frac{\partial H}{\partial y_i} \frac{dp^*_\beta}{d\beta} + (p - \beta) \left( \frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j} \right) \frac{dy^*_\beta}{d\beta} = -(1 - \frac{\partial H}{\partial y_i}).$$

Using Cramer’s rule, we have

$$\frac{dp^*_\beta}{d\beta} = \begin{vmatrix} 0 & \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} & \frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j} \\ -(1 - \frac{\partial H}{\partial y_i}) & (p - \beta)(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}) & \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} \\ \frac{\partial H}{\partial y_i} & -1 - \frac{\partial H}{\partial y_i} & (p - \beta)(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}) \end{vmatrix},$$

$$\frac{dy^*_\beta}{d\beta} = \begin{vmatrix} \frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} & 0 & \frac{\partial \pi_i^d}{\partial p_i} + \sum_{j \neq i} \frac{\partial \pi_i^d}{\partial p_i \partial y_j} \\ \frac{\partial H}{\partial y_i} & -1 - \frac{\partial H}{\partial y_i} & \frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} \\ \frac{\partial H}{\partial y_i} & -1 - \frac{\partial H}{\partial y_i} & (p - \beta)(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}) \end{vmatrix}. $$

Note that $\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} < 0$ (Vives 1999), $\frac{\partial H}{\partial y_i} > 0$, $1 - \frac{\partial H}{\partial y_i} > 0$, $\frac{\partial^2 H}{\partial y_i^2} < 0$, $\frac{\partial H}{\partial y_i} < 0$, $\frac{\partial^2 H}{\partial y_i \partial y_j} < 0$, in addition, $\frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j} > 0$. Then $dp^*/d\beta > 0$ and $dy^*/d\beta > 0$.

(ii) Differentiating (2) and (3) with respect to $w$, we have
\[
\frac{\partial^2 \pi^d}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 \pi^d}{\partial p_i \partial p_j} \frac{dp^*}{dw} + (\frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j}) \frac{dy^*}{dw} = -\frac{\partial^2 \pi^d}{\partial p_i \partial w} \text{ and }
\]
\[
\frac{\partial H}{\partial w} \frac{dp^*}{dw} + (p - \beta)(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_j \partial y_j}) \frac{dp^*}{dw} = 1.
\]

Note that \(\frac{\partial^2 \pi^d}{\partial p_i \partial w} > 0\). It can be shown that the only combination that cannot hold is \(dp^*/dw < 0\) and \(dy^*/dw > 0\).

**Proof of Proposition 5.** With linear demand, \(w_i^* = c_i + \sum_{j \neq i}(p_j^* - c_j)\theta/(b + \theta)\), since \(p_j^*\) is unaffected by \(\theta\), \(dw_i^*/d\theta > 0\). By equation (9), \(d\beta_i^*/d\theta > 0\).

**Proof of Proposition 6.** Substituting \((w_i^*, \beta_i^*)\) and \((\overrightarrow{p}, \overrightarrow{y})\) into function (1), we have
\[
\pi_i = (p_i^* - w_i^*)[L_i(\overrightarrow{p}) - y_i^* \Pr(\epsilon_i > y_i^*)/\Pr(\epsilon_i < y_i^*) + E[\min\{\epsilon_i, y_i^*\}]/\Pr(\epsilon_i < y_i^*)].
\]
Notice that \(\pi_i = (p_i^* - c_i)[L_i(\overrightarrow{p}) - y_i^* c_i/(p_i^* - c_i) + p_i^* E[\min\{\epsilon_i, y_i^*\}]/(p_i^* - c_i)].\)

By equations (4)-(5), we have
\[
c_i/(p_i^* - c_i) = \Pr(\epsilon_i > y_i^*)/\Pr(\epsilon_i < y_i^*) \text{ and }
\]
\[
p_i^*/(p_i^* - c_i) = 1/\Pr(\epsilon_i < y_i^*).
\]
Then \(\pi_i/\pi_i^c = (p_i^* - w_i^*)/(p_i^* - c_i) = [p_i^* - c_i + \sum_{j \neq i}(p_j^* - c_j)L_j^{(i)}(\overrightarrow{p})/L_i^{(i)}(\overrightarrow{p})]/(p_i^* - c_i)
\]
\[
= 1 + \sum_{j \neq i} L_j^{(i)}(\overrightarrow{p})/L_i^{(i)}(\overrightarrow{p}) = 1 - (n - 1)\theta/(b + \theta).
\]

The second equality holds because \(p_i^* - c_i = p_j^* - c_j\) in a symmetric game, and the last equality holds for the linear demand function.