Appendix for Product Design for Life Cycle Mismatch

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Abstract

This document contains the derivations of the formulae and proofs for the paper by Bradley and Guerrero (2008) that is entitled "Product Design for Life Cycle Mismatch."

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1 Profit for Exponential Time to Part Obsolescence

To compute the expected, discounted profit under exponential times until part obsolescence, we compute the profit for a deterministic time until obsolescence, which we use as a building block for the subsequent stochastic analysis. Two cases must be computed, one in which part obsolescence occurs before the end of the growth phase, $0 \leq \tau \leq T_g$, and one where part obsolescence occurs in the decay phase, $\tau > T_g$.

1.1 Nondurable Design Profit for Deterministic Part Obsolescence

When part obsolescence occurs in the growth phase, $\tau \in (0, T_g]$, the profit until part obsolescence at time τ, discounted back to time zero is

$$\begin{aligned}
\int_0^{\tau} \left(\pi - c_N^1\right) Q_s e^{-rs} ds &= \int_0^{\tau} \left(\pi - c_N^1\right) q_0 e^{\beta_e s} e^{-rs} ds \\
&= \frac{\left(\pi - c_N^1\right) q_0}{(\beta_e - r)} \left. e^{(\beta_e - r)s} \right|_0^{\tau} \\
&= \frac{\left(\pi - c_N^1\right) q_0}{(\beta_e - r)} \left[e^{(\beta_e - r)\tau} - 1 \right].
\end{aligned}$$

For discounted profit after part obsolescence, $t \in [\tau, \infty)$, we compute the following, with a unit profit contribution of $\pi - c_N^2$:

$$\begin{aligned}
\int_{\tau}^{\infty} \left(\pi - c_N^2\right) Q_t e^{-rt} dt &= \int_{\tau}^{T_g} \left(\pi - c_N^2\right) q_0 e^{\beta_e t} e^{-rt} dt + \int_{T_g}^{\infty} \left(\pi - c_N^2\right) q_0 e^{\beta_e T_g} e^{\beta_\ell (t - T_g)} e^{-rt} dt - C_D \\
&= \int_{\tau}^{T_g} \left(\pi - c_N^2\right) q_0 e^{(\beta_e - r)t} dt + \int_{T_g}^{\infty} \left(\pi - c_N^2\right) q_0 e^{(\beta_e - \beta_\ell) T_g} e^{(\beta_\ell - r)t} dt - C_D \\
&= \left. \frac{\left(\pi - c_N^2\right) q_0 e^{(\beta_e - r)t}}{(\beta_e - r)} \right|_{\tau}^{T_g} + \left. \frac{\left(\pi - c_N^2\right) q_0 e^{(\beta_e - \beta_\ell) T_g} e^{(\beta_\ell - r)t}}{(\beta_\ell - r)} \right|_{T_g}^{\infty} - C_D \\
&= \frac{\left(\pi - c_N^2\right) q_0}{(\beta_e - r)} \left(e^{(\beta_e - r)T_g} - e^{(\beta_e - r)\tau} \right) - \frac{\left(\pi - c_N^2\right) q_0 e^{(\beta_e - \beta_\ell) T_g}}{(\beta_\ell - r)} e^{(\beta_\ell - r)T_g} - C_D \\
&= \frac{\left(\pi - c_N^2\right) q_0}{(\beta_e - r)} \left(e^{(\beta_e - r)T_g} - e^{(\beta_e - r)\tau} \right) - \frac{\left(\pi - c_N^2\right) q_0 e^{(\beta_e - r) T_g}}{(\beta_\ell - r)} - C_D \\
&= \left[\frac{1}{(\beta_e - r)} e^{(\beta_e - r)T_g} - \frac{1}{(\beta_e - r)} e^{(\beta_e - r)\tau} - \frac{e^{(\beta_e - r)T_g}}{(\beta_\ell - r)} \right] \left(\pi - c_N^2\right) q_0 - C_D \\
&= \left[\frac{\beta_\ell - \beta_e}{(\beta_\ell - r)(\beta_e - r)} e^{(\beta_e - r)T_g} - \frac{1}{(\beta_e - r)} e^{(\beta_e - r)\tau} \right] \left(\pi - c_N^2\right) q_0 - C_D.
\end{aligned}$$

We designate the sum of profit in these two results minus the fixed costs of the nondurable product, which include the discounted cost of mitigating part obsolescence and the cost of the initial design, as $\Pi_N^e(\tau)$, where the subscript N denotes a nondurable design and the superscript e denotes "early" obsolescence:

$$\begin{aligned}\Pi_N^e(\tau) &= \frac{(\beta_e-\beta_\ell)}{(\beta_e-r)(r-\beta_\ell)}\left(\pi-c_N^2\right)q_0e^{(\beta_e-r)T_g}+\frac{1}{(\beta_e-r)}\left(c_N^2-c_N^1\right)q_0e^{(\beta_e-r)\tau}\\ &\quad -\frac{1}{(\beta_e-r)}\left(\pi-c_N^1\right)q_0-e^{-r\tau}C_O-C_N. \end{aligned} \tag{1}$$

When part obsolescence occurs in the decay phase, $\tau\in(T_g,\infty)$, the computation for variable profit is made in three segments because the demand growth rate parameters change at T_g and because the unit profit contribution changes at τ:

$$\begin{aligned}\Pi_N^\ell(\tau) &= \int_0^{T_g}\left(\pi-c_N^1\right)Q_se^{-rs}ds+e^{-rT_g}\int_{T_g}^{\tau}\left(\pi-c_N^1\right)Q_se^{-r(s-T_g)}ds\\ &\quad +e^{-r\tau}\int_\tau^\infty\left(\pi-c_N^2\right)Q_se^{-r(s-t)}ds-e^{-r\tau}C_O-C_N,\end{aligned}$$

where the superscript ℓ denotes "late" obsolescence.

Computing the first segment, we find:

$$\int_0^{T_g}\left(\pi-c_N^1\right)Q_se^{-rs}ds=\frac{\left(\pi-c_N^1\right)q_0}{(\beta_e-r)}\left[e^{(\beta_e-r)T_g}-1\right].$$

Then, for the second segment, discounting profit back to time zero we find

$$\begin{aligned}e^{-rT_g}\int_{T_g}^{\tau}\left(\pi-c_N^1\right)Q_se^{-r(s-T_g)}ds &= e^{-rT_g}\int_{T_g}^{\tau}\left(\pi-c_N^1\right)Q_se^{-r(s-T_g)}ds\\ &= \frac{e^{(\beta_e-r)T_g}\left(\pi-c_N^1\right)q_0}{(\beta_\ell-r)}\left(e^{(\beta_\ell-r)(\tau-T_g)}-1\right),\end{aligned}$$

where the last line is due to the value $Q_{T_g}=q_0e^{\beta_eT_g}$. Computing the third segment, we find

$$e^{-rt}\int_\tau^\infty\left(\pi-c_N^2\right)Q_se^{-r(s-t)}ds=\frac{\left(\pi-c_N^2\right)q_0e^{(\beta_e-r)T_g}e^{(\beta_\ell-r)(\tau-T_g)}}{(r-\beta_\ell)}.$$

Putting these three profit components together and subtracting the discounted cost of mitigating part obsolescence yields

$$\begin{aligned}\Pi_N^\ell(\tau) &= \frac{\left(\pi-c_N^1\right)q_0}{(\beta_e-r)}\left[e^{(\beta_e-r)T_g}-1\right]+\frac{e^{(\beta_e-r)T_g}\left(\pi-c_N^1\right)q_0}{(\beta_\ell-r)}\left(e^{(\beta_\ell-r)(\tau-T_g)}-1\right)\\ &\quad +\frac{\left(\pi-c_N^2\right)q_0e^{(\beta_e-r)T_g}e^{(\beta_\ell-r)(\tau-T_g)}}{(r-\beta_\ell)}-e^{-r\tau}C_O-C_N.\end{aligned} \tag{2}$$

1.2 Nondurable Design Profit with Exponential Time to Part Obsolescence

The expression for profit under a nondurable design, Π_N, is computed with the alternate formula for Π_N that is displayed in the main paper, using $\Pi_N^e(\tau)$ and $\Pi_N^\ell(\tau)$ from (1) and (2), and using an exponentially distributed time until part obsolescence with density function $f(\tau) = \lambda e^{-\lambda\tau}$:

$$\begin{aligned}\Pi_N &= \int_0^{T_g} \left\{ \frac{(\beta_e - \beta_\ell)}{(\beta_e - r)(r - \beta_\ell)} \left(\pi - c_N^2\right) q_0 e^{(\beta_e - r)T_g} + \frac{1}{(\beta_e - r)} \left(c_N^2 - c_N^1\right) q_0 e^{(\beta_e - r)\tau} \right. \\ &\quad \left. - \frac{1}{(\beta_e - r)} \left(\pi - c_N^1\right) q_0 - e^{-r\tau} C_O - C_N \right\} \lambda e^{-\lambda\tau} d\tau \\ &\quad + \int_{T_g}^{\infty} \left\{ \frac{\left(\pi - c_N^1\right) q_0}{(\beta_e - r)} \left[e^{(\beta_e - r)T_g} - 1\right] + \frac{e^{(\beta_e - r)T_g} \left(\pi - c_N^1\right) q_0}{(\beta_\ell - r)} \left(e^{(\beta_\ell - r)(\tau - T_g)} - 1\right) \right. \\ &\quad \left. + \frac{\left(\pi - c_N^2\right) q_0 e^{(\beta_e - r)T_g} e^{(\beta_\ell - r)(\tau - T_g)}}{(r - \beta_\ell)} - e^{-r\tau} C_O - C_N \right\} \lambda e^{-\lambda\tau} d\tau. \qquad (3)\end{aligned}$$

Tedious calculus and algebra yields

$$\begin{aligned}\Pi_N &= q_0 \left\{ \frac{(\beta_\ell - \beta_e)}{(\beta_\ell - r - \lambda)(\beta_e - r - \lambda)} \right\} \left(c_N^2 - c_N^1\right) e^{(\beta_e - r - \lambda)T_g} - \frac{\lambda C_O}{(r + \lambda)} - C_N \\ &\quad + \frac{q_0}{(\beta_e - r)} \left\{ \frac{(\beta_e - \beta_\ell)\left(\pi - c_N^2\right)}{(r - \beta_\ell)} e^{(\beta_e - r)T_g} - \frac{\lambda\left(c_N^2 - c_N^1\right)}{(\beta_e - r - \lambda)} - \left(\pi - c_N^1\right) \right\}. \qquad (4)\end{aligned}$$

The differential expected profit between a durable and nondurable design, $\Delta = \Pi_D - \Pi_N$, is computed with (3) and Π_D as shown in the accompanying paper:

$$\begin{aligned}\Delta(\lambda, T_g) &= \frac{(\beta_e - \beta_\ell)}{(\beta_e - r)(r - \beta_\ell)} q_0 e^{(\beta_e - r)T_g} \left(c_N^2 - c_D\right) + \left(c_D - c_N^1\right) \frac{q_0}{(\beta_e - r)} \\ &\quad - q_0 \left[\frac{(\beta_\ell - \beta_e)}{(\beta_\ell - r - \lambda)(\beta_e - r - \lambda)} \right] \left(c_N^2 - c_N^1\right) e^{(\beta_e - r - \lambda)T_g} \\ &\quad + \left\{ \frac{\lambda\left(c_N^2 - c_N^1\right)}{(\beta_e - r - \lambda)} \frac{q_0}{(\beta_e - r)} \right\} + \frac{\lambda C_O}{(r + \lambda)} - (C_D - C_N). \qquad (5)\end{aligned}$$

2 Proof of Theorem 1

Cumulatively, the following propositions describe the fundamental characteristics of the level set $\mathcal{L}(\lambda, T_g)$ that are described in Theorem 1 of the accompanying paper. Specifically, they describe the optimality regions for durable and nondurable designs when the time until part obsolescence is distributed according to an exponentially distribution and the product has high growth, $\beta_e > r$. In these propositions, we use $T_g^{\min}(\lambda)$ to denote the growth-phase duration, T_g, that minimizes $\Delta(\lambda, T_g)$, and thus maximizes the advantage of a nondurable design relative to a durable design for a given "arrival rate" of part obsolescence λ:

$$T_g^{\min}(\lambda) = \arg\min_{T_g} \Delta(\lambda, T_g).$$

Proposition 1. A unique, finite $T_g^{\min}(\lambda)$ exists for every $\lambda \in (0, \infty)$.

Proof.

The derivative of $\Pi_D - \Pi_N$ w.r.t. T_g is

$$\frac{d\Delta(\lambda, T_g)}{dT_g} = q_0(\beta_e - \beta_\ell) e^{(\beta_e - r)T_g} \left\{ \frac{c_N^2 - c_D}{r - \beta_\ell} + \frac{(c_N^2 - c_N^1) e^{-\lambda T_g}}{\beta_\ell - r - \lambda} \right\}, \tag{6}$$

which when evaluated at $T_g = 0$,

$$\left.\frac{d\Delta(\lambda, T_g)}{dT_g}\right|_{T_g=0} = \frac{q_0(\beta_e - \beta_\ell)}{\beta_\ell - r - \lambda} \left\{ \left(c_D - c_N^1\right) - \frac{\lambda(c_N^2 - c_D)}{r - \beta_\ell} \right\},$$

is negative for small $\lambda < \widetilde{\lambda}$, where $\widetilde{\lambda} = (r - \beta_\ell)(c_D - c_N^1)/(c_N^2 - c_D)$, and zero or positive for remaining values of λ. The expression in the braces of equation (6) is monotonic and increasing. If $\lambda < \widetilde{\lambda}$, then the derivative at $T_g = 0$ is initially negative and will eventually become positive as $T_g \to \infty$ such that a zero derivative, and the minimum of $\Pi_D - \Pi_N$, will exist at some $T_g > 0$, namely, at the root of

$$\frac{1}{(r - \beta_\ell)}\left(c_N^2 - c_D\right) + \frac{1}{(\beta_\ell - r - \lambda)}\left(c_N^2 - c_N^1\right) e^{-\lambda T_g} = 0.$$

If $\lambda \geq \widetilde{\lambda}$, then $\frac{d(\Pi_D - \Pi_N)}{dT_g} > 0$ for all $T_g \geq 0$ and, thus, $T_g^{\min}(\lambda) = 0$. Thus, the solution is

$$T_g^{\min}(\lambda) = \begin{cases} -\frac{1}{\lambda} \ln\left[\frac{(\beta_\ell - r - \lambda)}{(\beta_\ell - r)} \frac{\left(c_N^2 - c_D\right)}{\left(c_N^2 - c_N^1\right)} \right] & \lambda < \widetilde{\lambda} \\ 0 & \lambda \geq \widetilde{\lambda} \end{cases},$$

where $\widetilde{\lambda} = (r - \beta_\ell)(c_D - c_N^1)/(c_N^2 - c_D)$. ■

Proposition 2. The differential profit at $T_g^{\min}(\lambda)$, $\Delta\left(\lambda, T_g^{\min}(\lambda)\right)$, is increasing in λ. Moreover,

$$\lim_{\lambda\to\infty} \Delta\left(\lambda, T_g^{\min}(\lambda)\right) > 0,$$

such that a durable design is optimal as $\lambda \to \infty$ (i.e., as $\mathbf{E}\tau \to 0$) when $C_D - C_N < \frac{q_0\left(c_N^2 - c_D\right)}{r-\beta_\ell} + C_O$.

Proof.

We first evaluate $\Delta(\lambda, T_g)$ at $T_g^{\min}(\lambda)$ for λ small, that is $\lambda < \widetilde{\lambda}$:

$$\begin{aligned}\Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)} &= \frac{\lambda q_0\left(c_N^2 - c_D\right)}{\left(\beta_e - r\right)\left(\beta_e - r - \lambda\right)}\left\{1 - \frac{\beta_e - \beta_\ell}{r - \beta_\ell}\left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r}\frac{c_N^2 - c_D}{c_N^2 - c_N^1}\right]^{-\frac{(\beta_e - r)}{\lambda}}\right\} \\ &\quad + \frac{q_0\left(c_D - c_N^1\right)}{\beta_e - r - \lambda} + \frac{\lambda C_O}{r+\lambda} - \left(C_D - C_N\right).\end{aligned}$$

We find that the derivative of this expression w.r.t. λ (after tedious mathematics) is

$$\begin{aligned}\frac{d\,\Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)}}{d\lambda} &= \frac{q_0}{\left(\beta_e - r - \lambda\right)^2}\left\{\left(c_N^2 - c_N^1\right) - \left(c_N^2 - c_D\right)\left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r}\frac{c_N^2 - c_D}{c_N^2 - c_N^1}\right]^{-\frac{(\beta_e - r)}{\lambda}}\right. \\ &\quad \left.\times\frac{\beta_e - \beta_\ell}{r - \beta_\ell}\left\{1 + \frac{\beta_e - r - \lambda}{\beta_\ell - r - \lambda} + \frac{\beta_e - r - \lambda}{\lambda}\ln\left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r}\frac{c_N^2 - c_D}{c_N^2 - c_N^1}\right]\right\}\right\} \\ &\quad + \frac{rC_O}{(r+\lambda)^2}.\end{aligned}$$

Rewrite this as

$$\frac{d\,\Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)}}{d\lambda} = \frac{q_0}{\left(\beta_e - r - \lambda\right)^2}\left\{\left(c_N^2 - c_N^1\right) - \left(c_N^2 - c_D\right)\varphi(\lambda)\right\} + \frac{rC_O}{(r+\lambda)^2}. \qquad (7)$$

using $\varphi(\lambda)$ as a function of λ to denote

$$\begin{aligned}\varphi(\lambda) &= \left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r}\frac{\left(c_N^2 - c_D\right)}{c_N^2 - c_N^1}\right]^{-\frac{(\beta_e - r)}{\lambda}}\frac{\beta_e - \beta_\ell}{r - \beta_\ell} \\ &\quad \times\left\{1 + \frac{\beta_e - r - \lambda}{\beta_\ell - r - \lambda} + \frac{\beta_e - r - \lambda}{\lambda}\ln\left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r}\frac{c_N^2 - c_D}{c_N^2 - c_N^1}\right]\right\}.\end{aligned}$$

The derivative of $\varphi(\lambda)$ w.r.t. λ is

$$\begin{aligned}\frac{d\varphi(\lambda)}{d\lambda} &= \left[\frac{\left(\beta_\ell - r - \lambda\right)}{\left(\beta_\ell - r\right)}\frac{\left(c_N^2 - c_D\right)}{\left(c_N^2 - c_N^1\right)}\right]^{-\frac{(\beta_e - r)}{\lambda}}\frac{\left(\beta_e - \beta_\ell\right)\left(\beta_e - r - \lambda\right)}{\left(r - \beta_\ell\right)} \\ &\quad \times\left\{\frac{\left(\beta_e - r\right)}{\lambda}\left\{\frac{1}{\left(\beta_\ell - r - \lambda\right)} + \frac{1}{\lambda}\ln\left[\frac{\left(\beta_\ell - r - \lambda\right)}{\left(\beta_\ell - r\right)}\frac{\left(c_N^2 - c_D\right)}{\left(c_N^2 - c_N^1\right)}\right]\right\}^2 + \frac{1}{\left(\beta_\ell - r - \lambda\right)^2}\right\},\end{aligned}$$

which is positive for $\beta_e - r - \lambda > 0$ ($\lambda < \beta_e - r$) and negative for $\beta_e - r - \lambda < 0$ ($\lambda > \beta_e - r$). So, $\varphi(\lambda)$ attains its maximum w.r.t. λ at $\lambda = \beta_e - r$, which we evaluate to be

$$\max_{\lambda} \varphi(\lambda) = \left[\frac{(c_N^2 - c_D)}{(c_N^2 - c_N^1)}\right]^{-1}.$$

Consequently, the term within the braces in (7) attains a minimum at $\lambda = \beta_e - r$ of zero, implying that the derivative of $\Delta(\lambda, T_g)$ w.r.t. λ is always nonnegative.

Evaluating the limit of $\Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)}$ as $\lambda \to \infty$, we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \left\{ \Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)} \right\} &= \frac{q_0 (c_N^2 - c_D)}{\beta_e - r} \times \lim_{\lambda\to\infty} \left[\frac{\lambda}{\beta_e - r - \lambda}\right] \\
&\quad \times \lim_{\lambda\to\infty} \left\{ 1 - \frac{\beta_e - \beta_\ell}{r - \beta_\ell} \left[\frac{\beta_\ell - r - \lambda}{\beta_\ell - r} \frac{(c_N^2 - c_D)}{(c_N^2 - c_N^1)}\right]^{-\frac{(\beta_e - r)}{\lambda}} \right\} \\
&\quad + \lim_{\lambda\to\infty} \left[\frac{q_0 (c_D - c_N^1)}{\beta_e - r - \lambda}\right] + \lim_{\lambda\to\infty} \left[\frac{\lambda C_O}{r + \lambda}\right] - (C_D - C_N) \\
&= -\frac{q_0 (c_N^2 - c_D)}{\beta_e - r} \\
&\quad \times \lim_{\lambda\to\infty} \left\{ 1 - \frac{\beta_e - \beta_\ell}{r - \beta_\ell} e^{-\frac{(\beta_e - r)}{\lambda} \ln\left[\frac{(\beta_\ell - r - \lambda)}{(\beta_\ell - r)} \frac{(c_N^2 - c_D)}{(c_N^2 - c_N^1)}\right]} \right\} + C_O - (C_D - C_N).
\end{aligned}$$

Evaluating the exponent of the exponential function using l'Hôpital's rule, we obtain

$$\lim_{\lambda\to\infty} \left\{ -\frac{(\beta_e - r) \ln\left[\frac{(\beta_\ell - r - \lambda)}{(\beta_\ell - r)} \frac{(c_L^2 - c_D)}{(c_L^2 - c_L^1)}\right]}{\lambda} \right\} = \lim_{\lambda\to\infty} \left\{ \frac{\beta_e - r}{\beta_\ell - r - \lambda} \right\} = 0,$$

so that

$$\begin{aligned}
\lim_{\lambda\to\infty} \left\{ \Delta(\lambda, T_g)|_{T_g^{\min}(\lambda)} \right\} &= -\frac{q_0 (c_N^2 - c_D)}{(\beta_e - r)} \left(1 - \frac{\beta_e - \beta_\ell}{r - \beta_\ell}\right) + C_O - (C_D - C_N) \\
&= \frac{q_0 (c_N^2 - c_D)}{r - \beta_\ell} + C_O - (C_D - C_N),
\end{aligned}$$

which is positive when $C_D - C_N < \frac{q_0 (c_N^2 - c_D)}{r - \beta_\ell} + C_O$. ■

Proposition 3. A unique, finite λ exists such that $T_g^{\min}(\lambda) = 0$, which we denote by $\overline{\lambda}$, if $C_D - C_N \le \frac{q_0 (c_N^2 - c_D)}{r - \beta_\ell} + C_O$ A durable design is optimal for all T_g, for $\lambda > \overline{\lambda}$.

Proof. The proof follows immediately from Proposition 2. ■

Proposition 4. For $\beta_e - r \geq 0$, a durable design dominates a nondurable design as the growth-phase duration grows large, i.e.,

$$\lim_{T_g \to \infty} \Delta\left(\lambda, T_g\right) > 0.$$

Proof.

The value of equation (5) is dominated by the terms in T_g as T_g grows large. If $\lambda \geq \beta_e - r$, then the dominant term of (5) is

$$\frac{\beta_e - \beta_\ell}{\left(\beta_e - r\right)\left(r - \beta_\ell\right)} q_0 e^{\left(\beta_e - r\right)T_g}\left(c_N^2 - c_D\right),$$

which is positive and goes to infinity as $T_g \to \infty$. If $\lambda < \beta_e - r$, then the value of (5) is dominated by two terms,

$$\frac{\left(c_N^2 - c_D\right)\left(\beta_e - \beta_\ell\right)}{\left(\beta_e - r\right)\left(r - \beta_\ell\right)} q_0 e^{\left(\beta_e - r\right)T_g} - \frac{\left(c_N^2 - c_N^1\right)\left(\beta_\ell - \beta_e\right)}{\left(\beta_\ell - r - \lambda\right)\left(\beta_e - r - \lambda\right)} q_0 e^{\left(\beta_e - r - \lambda\right)T_g},$$

which, together, are also positive and go to infinity as $T_g \to \infty$. ■

References

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