Appendix 1

Claim: Π is strictly and jointly concave in q_1, q_2, q_3.

Proof: In order to show that Π is strictly and jointly concave in q_1, q_2, q_3, it is necessary to show that the determinants of the hessian (defined below) alternate in sign. Now given that:

$$\begin{aligned}\Pi &= q_1(d_1 - q_1 - r_{13}q_3) + q_2(d_2 - q_2 - r_{23}q_3) + \\ &\quad +q_3(d_3 - q_3 - r_{13}q_1 - r_{23}q_2)\end{aligned}$$

the hessian and its determinants are:

$$H = \begin{bmatrix} \frac{\partial^2\Pi}{\partial q_1^2} & \frac{\partial^2\Pi}{\partial q_1 \partial q_2} & \frac{\partial^2\Pi}{\partial q_1 \partial q_3} \\ \frac{\partial^2\Pi}{\partial q_2 \partial q_1} & \frac{\partial^2\Pi}{\partial q_2^2} & \frac{\partial^2\Pi}{\partial q_2 \partial q_3} \\ \frac{\partial^2\Pi}{\partial q_3 \partial q_1} & \frac{\partial^2\Pi}{\partial q_3 \partial q_2} & \frac{\partial^2\Pi}{\partial q_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2r_{13} \\ 0 & -2 & -2r_{23} \\ -2r_{13} & -2r_{23} & -2 \end{bmatrix}$$

$$|H_1^1| = |H_2^1| = |H_3^1| = -2 < 0$$

$$|H_{12}^2| = 4 > 0,\ |H_{13}^2| = 4(1 - r_{13}^2) > 0,\ |H_{23}^2| = 4(1 - r_{23}^2) > 0$$

$$|H_{123}^3| = -8(1 - r_{13}^2 - r_{23}^2) < 0 \text{ by assumption.}$$

Since the determinants of the hessian alternate in sign, we conclude that Π is strictly and jointly concave in q_1, q_2, q_3. □

Appendix 2

Theorem 1: The optimal product portfolio strategy for the firm can be identified as follows.

1. If $d_3 \in (0, \tau_1]$, the optimal product portfolio strategy is NMFPS;
2. If $d_3 \in (\tau_1, \tau_2]$, the optimal strategy is APS.
3. If $d_3 \in (\tau_2, \tau_3)$, and
   - If $\frac{d_2}{r_{23}} \leq \frac{d_1}{r_{13}}$, then the optimal strategy is PMFPS1; and
   - If $\frac{d_2}{r_{23}} > \frac{d_1}{r_{13}}$, then the optimal strategy is PMFPS2.
4. If $d_3 \in [\tau_3, \infty)$, then the optimal strategy is SMFPS.

where:

$$\begin{aligned}
\tau_1 &= r_{13}d_1 + r_{23}d_2 \\
\tau_2 &= \min\{(\frac{1-r_{23}^2}{r_{13}})d_1 + r_{23}d_2, r_{13}d_1 + (\frac{1-r_{13}^2}{r_{23}})d_2\} \\
\tau_3 &= \max\{(\frac{1}{r_{13}})d_1, (\frac{1}{r_{23}})d_2\}
\end{aligned}$$

Proof: Given the strict concavity of Π (see Appendix 1), it is necessary and sufficient to set the FOC equal to 0 to determine the optimal quantities of each product (i.e., q_i^* for $i = 1, 2, 3$) which should be offered by the firm. In addition, the results shown in Table 2 and the definitions of τ_1 and τ_2 above provide us with the following guidelines for when each strategy is feasible:

- $0 < d_3 < \infty \Rightarrow$ NMFPS and SMFPS are both feasible.
- $\tau_1 \leq d_3 \leq \tau_2 \Rightarrow$ APS is feasible.
- $r_{13}d_1 < d_3 < r_{13}^{-1}d_1 \Rightarrow$ PMFPS1 is feasible.
- $r_{23}d_2 < d_3 < r_{23}^{-1}d_2 \Rightarrow$ PMFPS2 is feasible.

The remainder of this proof is provided depending upon the range of values for the parameter d_3 in the Theorem.

Case 1: $d_3 \in (0, \tau_1]$

To start with, it is obvious that since $r_{13}d_1 + r_{23}d_2 > r_{13}d_1$ and $r_{13}d_1 + r_{23}d_2 > r_{23}d_2$, in the range $0 < d_3 < r_{13}d_1 + r_{23}d_2$, the potentially feasible strategies are NMFPS, SMFPS, PMFPS1, and PMFPS2. Keeping in mind our assumption of $r_{13}^2 + r_{13}^2 < 1$ which implies that $1 - r_{13}^2 > r_{23}^2$ and $1 - r_{23}^2 > r_{13}^2$, let us examine the differences in profits between the feasible strategies.

$$\begin{aligned}
\Pi_{NMFPS} - \Pi_{PMFPS1} &= 0.25[(d_1^2 + d_2^2 - y(d_1^2 + d_3^2 - 2r_{13}d_1d_3)] \\
&= 0.25y[d_2^2(1 - r_{13}^2) - (d_1r_{13} - d_3)^2] \\
&> 0.25y[d_2^2r_{23}^2 - (d_1r_{13} - d_3)^2] \qquad \text{since } 1 - r_{13}^2 > r_{23}^2 \\
&= 0.25y[(d_2r_{23} - d_1r_{13} + d_3)(d_2r_{23} + d_1r_{13} - d_3) \\
&\geq 0
\end{aligned}$$

This last statement is true since: (a) $d_3 - d_1r_{13} \geq 0$ which is a feasibility condition for PMFPS1; and (b) $d_2r_{23} + d_1r_{13} - d_3 \geq 0$ which is the range for the parameter d_3 we are investigating. Hence, we can conclude that NMFPS is preferred over PMFPS1. In a similar manner it is possible to show that $\Pi_{NMFPS} - \Pi_{PMFPS2} > 0$ and thus, NMFPS is also preferred over PMFPS2.

Now in the range $0 < d_3 \leq r_{13}d_1 + r_{23}d_2$, we know that $\Pi_{SMFPS} = d_3^2$ is monotonically increasing. Thus, it achieves its maximum when $d_3 = r_{13}d_1 + r_{23}d_2$ and hence, let us consider:

$$\begin{aligned}
& \Pi_{SMFPS}(d_3 = r_{13}d_1 + r_{23}d_2) - \Pi_{NMFPS} \\
= \; & (d_1r_{13} + d_2r_{23})^2 - (d_1^2 + d_2^2) \\
= \; & d_1^2r_{13}^2 + d_2^2r_{23}^2 + 2d_1d_2r_{13}r_{23} - (d_1^2 + d_2^2) \\
= \; & -(d_1^2 + d_2^2)(1 - r_{13}^2 - r_{23}^2) + (2d_1d_2r_{13}r_{23} - d_1^2r_{23}^2 - d_2^2r_{13}^2) \\
= \; & -(d_1^2 + d_2^2)(1 - r_{13}^2 - r_{23}^2) - (d_1r_{23} - d_2r_{13})^2 \\
< \; & 0
\end{aligned}$$

As a result, when $0 < d_3 < r_{13}d_1 + r_{23}d_2$ we know that the profits under NMFPS dominate the profits under SMFPS, PMFPS1, and PMFPS2. Hence, in this range, the preferred strategy is NMFPS.

Case 2: $d_3 \in (\tau_1, \tau_2]$

In this range, the solution provided by APS is feasible. Given that this solution is globally optimal for our problem (since Π is strictly concave - see Appendix 1), it is obvious that APS would dominate all other potentially feasible strategies for this range.

Case 3: $d_3 \in (\tau_2, \tau_3)$ or
$\min\{r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23},\, r_{23}^{-1}d_2(1-r_{13}^2)+d_1r_{13}\} < d_3 < \max\{r_{13}^{-1}d_1, r_{23}^{-1}d_2\}$

In general, PMFPS1, PMFPS2, NMFPS and SMFPS are all feasible strategies in this range. We consider two separate sub-cases to identify the dominant strategy.

Case 3A: $r_{13}^{-1}d_1 \leq r_{23}^{-1}d_2$
In this case,

$$r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23}-(r_{23}^{-1}d_2(1-r_{13}^2)+d_1r_{13}) = (1-r_{13}^2-r_{23}^2)[r_{13}^{-1}d_1-r_{23}^{-1}d_2] < 0$$

This implies that the range specified in Case 3, can be restated as $r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23} < d_3 < r_{23}^{-1}d_2$. In this range, PMFPS1 is infeasible since $r_{13}^{-1}d_1 - (r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23}) = r_{23}^2(r_{13}^{-1}d_1 - r_{23}^{-1}d_2) < 0$. Thus, under Case 3A, the feasible strategies are PMFPS2, NMFPS, and SMFPS. Comparing profits for these strategies:

$$\Pi_{PMFPS2} - \Pi_{SFMPS} = 0.25z[(d_2 - r_{23}d_3)^2] > 0$$

Now it is easy to show that Π_{PMFPS2} is monotonically increasing in the range for d_3 given by Case 3A. Thus, the profits under PMFPS2 are minimum when $d_3 = r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23}+\epsilon$ where ϵ is set to be sufficiently small. Say $\epsilon \approx 0$, then consider:

$$\begin{aligned}
&\Pi_{PMFPS2}(d_3 = r_{13}^{-1}d_1(1-r_{23}^2)+d_2r_{23}) - \Pi_{NFMPS} \\
&= d_2^2 + (1-r_{23}^2)^{-1}(d_3 - r_{23}d_2)^2 - (d_1^2 + d_2^2) \\
&= (1-r_{23}^2)^{-1}(r_{13}^{-1}d_1(1-r_{23}^2))^2 - d_1^2 \\
&= r_{13}^{-2}d_1^2(1-r_{23}^2) - d_1^2 > 0 \qquad \text{since } 1-r_{23}^2 > r_{13}^2 \Rightarrow r_{13}^{-2}(1-r_{23}^2) > 1
\end{aligned}$$

Given these results, we can conclude that PMFPS2 is the dominant strategy for Case 3A.

Case 3B: $r_{13}^{-1}d_1 > r_{23}^{-1}d_2$

In this case,

$$r_{13}^{-1}d_1(1 - r_{23}^2) + d_2 r_{23} - (r_{23}^{-1}d_2(1 - r_{13}^2) + d_1 r_{13}) = (1 - r_{13}^2 - r_{23}^2)[r_{13}^{-1}d_1 - r_{23}^{-1}d_2] > 0$$

This implies that the range specified in Case 3, can be restated as $r_{23}^{-1}d_2(1 - r_{13}^2) + d_1 r_{13} < d_3 < r_{13}^{-1}d_1$. In this range, PMFPS2 is infeasible since $r_{23}^{-1}d_2 - (r_{23}^{-1}d_2(1 - r_{13}^2) + d_1 r_{13}) = r_{13}^2(r_{23}^{-1}d_2 - r_{13}^{-1}d_1) < 0$. Thus, under Case 3B, the feasible strategies are PMFPS1, NMFPS, and SMFPS. Comparing profits for these strategies:

$$\Pi_{PMFPS1} - \Pi_{SFMPS} = 0.25y[(d_1 - r_{13}d_3)^2] > 0$$

Now it is easy to show that Π_{PMFPS1} is monotonically increasing in the range for d_3 given by Case 3B. Thus, the profits under PMFPS1 are minimum when $d_3 = r_{23}^{-1}d_2(1 - r_{13}^2) + d_1 r_{13} + \epsilon$ where ϵ is set to be sufficiently small. Say $\epsilon \approx 0$, then as with Case 3A, it can be shown that:

$$\Pi_{PMFPS1}(d_3 = r_{23}^{-1}d_2(1 - r_{13}^2) + d_1 r_{13}) - \Pi_{NFMPS} > 0$$

Given these results, we can conclude that PMFPS1 is the dominant strategy for Case 3B.

Case 4: $d_3 \in [\tau_3, \infty)$

When $r_{13}^{-1}d_1 \leq r_{23}^{-1}d_2$

$$\begin{aligned}
& \Pi_{SMFPS}(d_3 = r_{23}^{-1}d_2) - \Pi_{NFMPS} \\
= \; & r_{23}^{-2}d_2 - (d_1^2 + d_2^2) \\
= \; & r_{23}^{-2}d_2(1 - r_{23}^2) - d_1^2 \\
> \; & r_{23}^{-2}d_2 r_{13}^2 - d_1^2 \qquad \text{since } 1 - r_{23}^2 > r_{13}^2 \\
> \; & 0 \qquad \text{since } r_{13}^{-1}d_1 < r_{23}^{-1}d_2 \Rightarrow r_{23}^{-1}d_2 r_{13} > d_1
\end{aligned}$$

Similarly, when $r_{13}^{-1}d_1 > r_{23}^{-1}d_2$ $\Pi_{SMFPS}(d_3 = r_{13}^{-1}d_1) - \Pi_{NFMPS} > 0$. Let $A = \max \{r_{13}^{-1}d_1, r_{23}^{-1}d_2\}$, it is obvious that $\Pi_{SMFPS}(d_3 = x) > \Pi_{SMFPS}(d_3 = A)$ for $\forall x > A$. Since PMFPS1 and PMFPS2 are infeasible in this region, SMFPS is the only dominant strategy when $\max \{r_{13}^{-1}d_1, r_{23}^{-1}d_2\} \leq d_3$.