Claim: $\Pi$ is strictly and jointly concave in $q_1, q_2, q_3$.

Proof: In order to show that $\Pi$ is strictly and jointly concave in $q_1, q_2, q_3$, it is necessary to show that the determinants of the hessian (defined below) alternate in sign. Now given that:

$$
\Pi = q_1(d_1 - q_1 - r_{13} q_3) + q_2(d_2 - q_2 - r_{23} q_3) + 
+ q_3(d_3 - q_3 - r_{13} q_1 - r_{23} q_2)
$$

the hessian and its determinants are:

$$
H = 
\begin{bmatrix}
\frac{\partial^2 \Pi}{\partial q_1^2} & \frac{\partial^2 \Pi}{\partial q_1 \partial q_2} & \frac{\partial^2 \Pi}{\partial q_1 \partial q_3} \\
\frac{\partial^2 \Pi}{\partial q_2 \partial q_1} & \frac{\partial^2 \Pi}{\partial q_2^2} & \frac{\partial^2 \Pi}{\partial q_2 \partial q_3} \\
\frac{\partial^2 \Pi}{\partial q_3 \partial q_1} & \frac{\partial^2 \Pi}{\partial q_3 \partial q_2} & \frac{\partial^2 \Pi}{\partial q_3^2}
\end{bmatrix}
= 
\begin{bmatrix}
-2 & 0 & -2r_{13} \\
0 & -2 & -2r_{23} \\
-2r_{13} & -2r_{23} & -2
\end{bmatrix}
$$

$$
|H^1_1| = |H^2_2| = |H^3_3| = -2 < 0
$$

$$
|H^1_2| = 4 > 0, \quad |H^2_{13}| = 4(1 - r_{13}^2) > 0, \quad |H^2_{23}| = 4(1 - r_{23}^2) > 0
$$

$$
|H^3_{123}| = -8(1 - r_{13}^2 - r_{23}^2) < 0 \text{ by assumption.}
$$

Since the determinants of the hessian alternate in sign, we conclude that $\Pi$ is strictly and jointly concave in $q_1, q_2, q_3$. □
Appendix 2

**Theorem 1**: The optimal product portfolio strategy for the firm can be identified as follows.

1. If $d_3 \in (0, \tau_1]$, the optimal product portfolio strategy is NMFPS;
2. If $d_3 \in (\tau_1, \tau_2]$, the optimal strategy is APS.
3. If $d_3 \in (\tau_2, \tau_3)$, and
   - If $\frac{d_2}{r_2} \leq \frac{d_1}{r_{13}}$, then the optimal strategy is PMFPS1; and
   - If $\frac{d_2}{r_2} > \frac{d_1}{r_{13}}$, then the optimal strategy is PMFPS2.
4. If $d_3 \in [\tau_3, \infty)$, then the optimal strategy is SMFPS.

where:

$$
\begin{align*}
\tau_1 & = r_{13}d_1 + r_{23}d_2 \\
\tau_2 & = \min\{(\frac{1 - r_{23}^2}{r_{13}})d_1 + r_{23}d_2, r_{13}d_1 + (\frac{1 - r_{13}^2}{r_{23}})d_2\} \\
\tau_3 & = \max\{(\frac{1}{r_{13}})d_1, (\frac{1}{r_{23}})d_2\}
\end{align*}
$$

**Proof**: Given the strict concavity of $\Pi$ (see Appendix 1), it is necessary and sufficient to set the FOC equal to 0 to determine the optimal quantities of each product (i.e., $q_i^*$ for $i = 1, 2, 3$) which should be offered by the firm. In addition, the results shown in Table 2 and the definitions of $\tau_1$ and $\tau_2$ above provide us with the following guidelines for when each strategy is feasible:

- $0 < d_3 < \infty \Rightarrow$ NMFPS and SMFPS are both feasible.
- $\tau_1 \leq d_3 \leq \tau_2 \Rightarrow$ APS is feasible.
- $r_{13}d_1 < d_3 < r_{13}^{-1}d_1 \Rightarrow$ PMFPS1 is feasible.
- $r_{23}d_2 < d_3 < r_{23}^{-1}d_2 \Rightarrow$ PMFPS2 is feasible.
The remainder of this proof is provided depending upon the range of values for the parameter $d_3$ in the Theorem.

Case 1: $d_3 \in (0, \tau_1]

To start with, it is obvious that since $r_{13}d_1 + r_{23}d_2 > r_{13}d_1$ and $r_{13}d_1 + r_{23}d_2 > r_{23}d_2$, in the range $0 < d_3 < r_{13}d_1 + r_{23}d_2$, the potentially feasible strategies are NMFPS, SMFPS, PMFPS1, and PMFPS2. Keeping in mind our assumption of $r_{13}^2 + r_{13}^2 < 1$ which implies that $1 - r_{13}^2 > r_{23}^2$ and $1 - r_{23}^2 > r_{13}^2$, let us examine the differences in profits between the feasible strategies.

$$
\Pi_{NMFPS} - \Pi_{PMFPS1} = 0.25[(d_1^2 + d_2^2 - y)(d_1^2 + d_3^2 - 2r_{13}d_1d_3)]
\geq 0$$

This last statement is true since: (a) $d_3 - d_1r_{13} \geq 0$ which is a feasibility condition for PMFPS1; and (b) $d_2r_{23} + d_1r_{13} - d_3 \geq 0$ which is the range for the parameter $d_3$ we are investigating. Hence, we can conclude that NMFPS is preferred over PMFPS1. In a similar manner it is possible to show that $\Pi_{NMFPS} - \Pi_{PMFPS2} > 0$ and thus, NMFPS is also preferred over PMFPS2.

Now in the range $0 < d_3 \leq r_{13}d_1 + r_{23}d_2$, we know that $\Pi_{SMFPS} = d_3^2$ is monotonically increasing. Thus, it achieves its maximum when $d_3 = r_{13}d_1 + r_{23}d_2$ and hence, let us consider:

$$
\Pi_{SMFPS}(d_3 = r_{13}d_1 + r_{23}d_2) - \Pi_{NMFPS}
\geq (d_1r_{13} + d_2r_{23})^2 - (d_1^2 + d_2^2)
\geq d_1^2r_{13}^2 + d_2^2r_{23}^2 + 2d_1d_2r_{13}r_{23} - (d_1^2 + d_2^2)
= -(d_1^2 + d_2^2)(1 - r_{13}^2 - r_{23}^2) + (2d_1d_2r_{13}r_{23} - d_1^2r_{23}^2 - d_2^2r_{13}^2)
\geq -(d_1^2 + d_2^2)(1 - r_{13}^2 - r_{23}^2) - (d_1r_{23} - d_2r_{13})^2
< 0$$

As a result, when $0 < d_3 < r_{13}d_1 + r_{23}d_2$ we know that the profits under NMFPS dominate the
profits under SMFPS, PMFPS1, and PMFPS2. Hence, in this range, the preferred strategy is NMFPS.

**Case 2:** $d_3 \in (\tau_1, \tau_2]

In this range, the solution provided by APS is feasible. Given that this solution is globally optimal for our problem (since $\Pi$ is strictly concave - see Appendix 1), it is obvious that APS would dominate all other potentially feasible strategies for this range.

**Case 3:** $d_3 \in (\tau_2, \tau_3)$ or

$$\min \{r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23}, r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13}\} < d_3 < \max\{r_{13}^{-1}d_1, r_{23}^{-1}d_2\}$$

In general, PMFPS1, PMFPS2, NMFPS and SMFPS are all feasible strategies in this range. We consider two separate sub-cases to identify the dominant strategy.

**Case 3A:** $r_{13}^{-1}d_1 \leq r_{23}^{-1}d_2$

In this case,

$$r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23} - (r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13}) = (1 - r_{13}^2 - r_{23}^2)[r_{13}^{-1}d_1 - r_{23}^{-1}d_2] < 0$$

This implies that the range specified in Case 3, can be restated as $r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23} < d_3 < r_{23}^{-1}d_2$. In this range, PMFPS1 is infeasible since $r_{13}^{-1}d_1 - (r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23}) = r_{23}^2(r_{13}^{-1}d_1 - r_{23}^{-1}d_2) < 0$. Thus, under Case 3A, the feasible strategies are PMFPS2, NMFPS, and SMFPS. Comparing profits for these strategies:

$$\Pi_{PMFPS2} - \Pi_{SFMPS} = 0.25z[(d_2 - r_{23}d_3)^2] > 0$$

Now it is easy to show that $\Pi_{PMFPS2}$ is monotonically increasing in the range for $d_3$ given by Case 3A. Thus, the profits under PMFPS2 are minimum when $d_3 = r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23} + \epsilon$ where $\epsilon$ is set to be sufficiently small. Say $\epsilon \approx 0$, then consider:

$$\Pi_{PMFPS2}(d_3 = r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23}) - \Pi_{NMFPS}$$

$$= d_2 + (1 - r_{23}^2)^{-1}(d_3 - r_{23}d_2)^2 - (d_1^2 + d_2^2)$$

$$= (1 - r_{23}^2)^{-1}(r_{13}^{-1}d_1(1 - r_{23}^2))^2 - d_1^2$$

$$= r_{13}^{-2}d_1^2(1 - r_{23}^2) - d_1^2 > 0 \quad \text{since} \ 1 - r_{23}^2 > r_{13}^2 \Rightarrow r_{13}^{-2}(1 - r_{23}^2) > 1$$

Given these results, we can conclude that PMFPS2 is the dominant strategy for Case 3A.
Case 3B: $r_{13}^{-1}d_1 > r_{23}^{-1}d_2$

In this case,

$$r_{13}^{-1}d_1(1 - r_{23}^2) + d_2r_{23} - (r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13}) = (1 - r_{13}^2 - r_{23}^2)[r_{13}^{-1}d_1 - r_{23}^{-1}d_2] > 0$$

This implies that the range specified in Case 3, can be restated as $r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13} < d_3 < r_{13}^{-1}d_1$. In this range, PMFPS2 is infeasible since $r_{23}^{-1}d_2 - (r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13}) = r_{13}^2(r_{23}^{-1}d_2 - r_{13}^{-1}d_1) < 0$. Thus, under Case 3B, the feasible strategies are PMFPS1, NMFPS, and SMFPS. Comparing profits for these strategies:

$$\Pi_{PMFPS1} - \Pi_{SFMPS} = 0.25y[(d_1 - r_{13}d_3)^2] > 0$$

Now it is easy to show that $\Pi_{PMFPS1}$ is monotonically increasing in the range for $d_3$ given by Case 3B. Thus, the profits under PMFPS1 are minimum when $d_3 = r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13} + \epsilon$ where $\epsilon$ is set to be sufficiently small. Say $\epsilon \approx 0$, then as with Case 3A, it can be shown that:

$$\Pi_{PMFPS1}(d_3 = r_{23}^{-1}d_2(1 - r_{13}^2) + d_1r_{13}) - \Pi_{NMFPS} > 0$$

Given these results, we can conclude that PMFPS1 is the dominant strategy for Case 3B.

Case 4: $d_3 \in [\tau_3, \infty)$

When $r_{13}^{-1}d_1 \leq r_{23}^{-1}d_2$

$$\Pi_{SMFPS}(d_3 = r_{23}^{-1}d_2) - \Pi_{NMFPS} = r_{23}^{-2}d_2 - (d_1^2 + d_2^2) = r_{23}^{-2}d_2(1 - r_{23}^2) - d_1^2 > r_{23}^{-2}d_2r_{13}^2 - d_1^2 \quad \text{since} \quad 1 - r_{23}^2 > r_{13}^2 > 0$$

Similarly, when $r_{13}^{-1}d_1 > r_{23}^{-1}d_2$ $\Pi_{SMFPS}(d_3 = r_{13}^{-1}d_1) - \Pi_{NMFPS} > 0$. Let $A = \max\{r_{13}^{-1}d_1, r_{23}^{-1}d_2\}$, it is obvious that $\Pi_{SMFPS}(d_3 = x) > \Pi_{SMFPS}(d_3 = A)$ for $\forall x > A$. Since PMFPS1 and PMFPS2 are infeasible in this region, SMFPS is the only dominant strategy when $\max\{r_{13}^{-1}d_1, r_{23}^{-1}d_2\} \leq d_3$. 

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