

INTEGRATED DESIGN AND OPTIMIZATION MODELS FOR THE SIX SIGMA PROCESS

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ABSTRACT NUMBER: 002-0487

SECOND WORLD CONFERENCE ON POM

AND 15TH ANNUAL POM CONFERENCE, CANCUN, MEXICO

APRIL 30-MAY 3, 2004

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Abstract

A typical process for quality improvement has six phases: Define, Measure, Analyze, Improve, Control and Technology Transfer. (D)MAIC(T) is a well known and highly utilized process for Design for Six Sigma (DFSS). Several optimization models to improve and extend the methodology, with emphasis on the analysis and improvement phases are presented. In order to improve the process, the system transfer function and variance transmission equations are developed which are very important for the six sigma process. Ideas for future research for integrated design and optimization models for the Six Sigma process are given.

Keywords: Design for Six Sigma, Optimization, System Transfer Function, Variance Transmission Equation.

Introduction

Design for Six Sigma (DFSS) is a rigorous, data-driven, decision-making approach to analyze the root causes of problems and improve the process capability to the six sigma level. It utilizes a systematic six-phase, problem-solving process called (D)MAIC(T): Define, Measure, Analyze, Improve, Control and Technology Transfer. The process of (D)MAIC(T) stays on track by establishing deliverables for each phase, by creating engineering models over time to reduce the process variation, and by continuously improving the predictability of system performance. Each of the six phases in the (D)MAIC(T) process is critical to achieve success.

0. Define - What problem needs to be solved?

It is important to define the scope, expectations, resources and timelines for the selected project. The definition phase for the Six Sigma approach identifies specifically the scope of the project, defines the customer and critical to quality (CTQ) issues from the viewpoint of the customer, and develops the core processes.

1. Measure - What is the current capability of the process?

Design for Six Sigma is a data-driven approach which requires quantifying and benchmarking the process using actual data. In this phase, the performance or process capability of the process for the critical to quality (CTQ) characteristics are evaluated.

2. Analyze – What are root causes for process variability?

Once the project is understood and the baseline performance is documented, it is time to do an analysis of the process. In this phase, Six Sigma approach applies statistical tools to validate root causes of problems. The objective is to understand the process at a level sufficient to be able to formulate options for improvement. We should be able to compare the various options with each

other to determine the most promising alternatives. In general, during the process of analysis, we analyze the data collected and use process maps to determine root causes of defects and prioritize opportunities for improvement.

3. Improve - How to improve the process capability?

During the improvement phase of Six Sigma approach, ideas and solutions are put into work to initialize the change. Based on the root causes discovered and validated for the existing opportunity, the target process is improved by designing creative solutions to fix and prevent problems. Some experiments and trials may be implemented in order to find the best solution. If a mathematical model is developed, then optimization methods are utilized to determine the optimum solution.

4. Control - What controls can be put in place to sustain the improvement?

The key to the overall success of Six Sigma methodology is its sustainability, which seeks to make everything incrementally better on a continuous basis. The sum of all these incremental improvements can be quite large. Without continuous sustenance, over time things will get worse until finally it is time for another effort towards improvement. As part of Six Sigma approach, performance tracking mechanisms and measurements are put in place to assure that the gains made in the project are not lost over time, and the process remains on the new course.

∞. Technology Transfer

Ideas and knowledge developed in one part of the organization can be transferred to other parts of the organization. In addition, the methods and solutions developed for one product or process can be applied to other similar products or processes. With the technology transfer, Six Sigma approach starts to create phenomenal returns.

There are many optimization problems in the six phases of this methodology. In this paper, we will review and develop the optimization models to improve the quality of the system to the six sigma level, utilizing the tools of probabilistic design, robust design, design of experiments, multivariable optimization, and simulation techniques. The goal is to investigate and explore the engineering, mathematical and statistical basis of (D)MAIC(T) process. We also want to improve and extend the methodology for the analysis and improvement phases of the Six Sigma process. In the analysis phase, the system transfer function and variance transmission equation need to be developed, so that we can formulate options for improvement by understanding the system. Based on the system transfer function or variance transmission equation, optimization models are formulated and solved to obtain the optimum decisions. The above topics will be discussed in details in the following sections.

System Transfer Function

A typical system consists of many input variables and one or multiple output variables. The input variables include both controllable factors and uncontrollable or noise factors. For a system with one output variable as given in Figure 1, $X_1, X_2, \dots, X_n$ are the controllable variables, and y is the realization of random output variable Y .

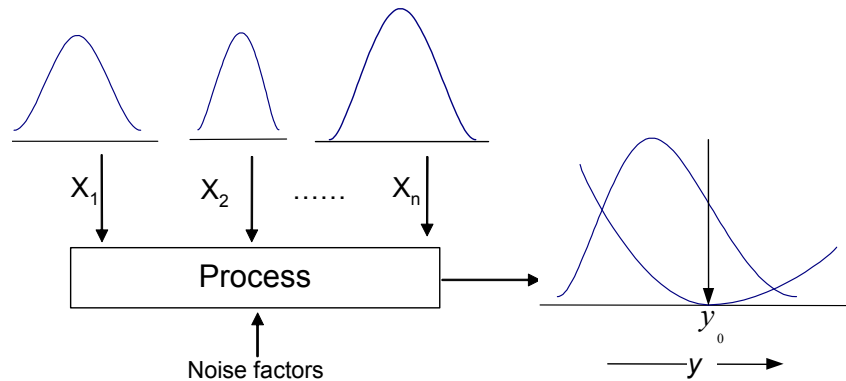


Figure 1. A General System Diagram

In the measurement phase of the (D)MAIC(T) process, the critical to quality (CTQ) characteristics are developed. In order to understand the system, we need to analyze the functional relationship between the output variable and the input variables, which can be described as a system transfer function or STF as below:

$$y = g(x_1, x_2, \dots, x_n) + \varepsilon \quad (1)$$

where ε is the system error caused by the noise factors. Let $y, x_1, x_2, \dots, x_n$ be the realization of random variables $Y, X_1, X_2, \dots, X_n$ respectively. The critical to quality (CTQ) characteristics in the system are linked together through the system transfer functions. The CTQ flow-down tree [2] in Figure 2 illustrates how the system transfer functions establish the relationships among the CTQs at different levels.

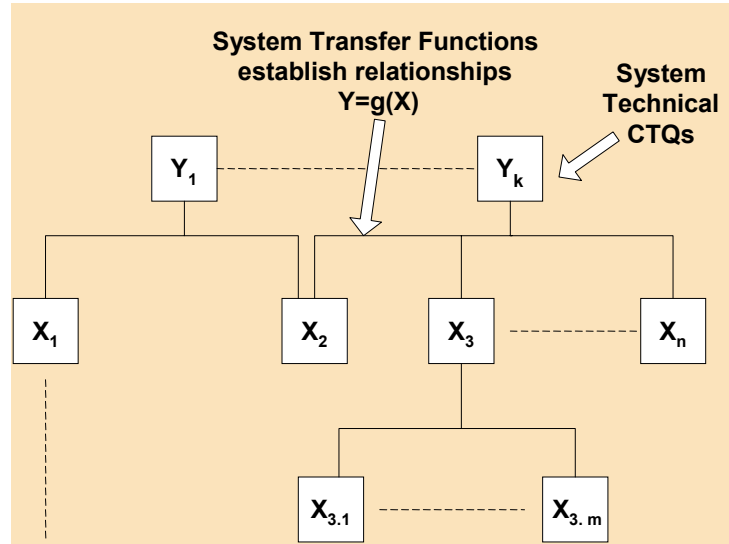


Figure 2. The CTQ Flow-Down Tree Diagram

For many complex systems, the analytical forms of the STF are unknown explicitly. Even if it is known, it is usually very complicated to work with. Given a set of values of input variables of the system, the corresponding values of response variables can be achieved through computer simulations or actual experiments. Based on the simulated or experimental data, an empirical

model of the system transfer function can be developed using the regression method [10]. Furthermore, the mean and variance models can be obtained by applying conditional expectation and variance operators to the regression model. Myers and Montgomery [10] discuss this approach to obtain the mean and variance response surfaces.

In many cases, the model is often easily and elegantly constructed as a series of orthogonal polynomials. Compare with other orthogonal functions, the orthogonal polynomials are particularly convenient for at least two reasons. First, polynomials are easier to work with than irrational or transcendental functions; second, the terms in orthogonal polynomials are statistically independent, which facilitates both their generation and processing. One of the other advantages of orthogonal polynomials is that users can simply develop their own system of functions in accordance with the particular problem. More often, a problem can be transformed to one of the standard families of polynomials, for which all significant relations have already been worked out.

Orthogonal polynomials can be used whether the values of controllable factors X 's are equally or unequally spaced. However, the computation is relatively easy when the values of factor levels are in equal steps. For a system with only one equal-step input variable X , the general orthogonal polynomial model of the functional relationship between response variable Y and X is given as

$$y = \mu + \alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) + \dots \quad (2)$$

where x is the value of factor level; y is the measured response; μ is the grand mean of all responses; and $P_k(x)$ is the k^{th} order orthogonal polynomial of factor X . The transformations for the powers of x into orthogonal polynomials $P_k(x)$ up to the cubic degree are given below:

$$\begin{aligned}
P_1(x) &= \lambda_1 \left[\frac{x - \bar{x}}{d} \right]; \quad P_2(x) = \lambda_2 \left[\left(\frac{x - \bar{x}}{d} \right)^2 - \left(\frac{t^2 - 1}{12} \right) \right]; \\
P_3(x) &= \lambda_3 \left[\left(\frac{x - \bar{x}}{d} \right)^3 - \left(\frac{x - \bar{x}}{d} \right) \left(\frac{3t^2 - 7}{20} \right) \right]
\end{aligned} \tag{3}$$

where $\bar{x}$ is the average value of factor levels; t is the number of levels of the factor; d is the distance between factor levels; the constant λ_k makes $P_k(x)$ an integral value for each x . The values of the orthogonal polynomials $P_k(x)$ have been tabulated up to $t=104$ [8].

Given the response y_i for the i^{th} level of X , x_i , $i = 1, 2, \dots, t$, the estimates of the α_k coefficients for the orthogonal polynomial equation (2) are calculated as $\alpha_k = \sum_{i=1}^t y_i P_k(x_i) / \sum_{i=1}^t P_k(x_i)^2$, for $k = 1, 2, \dots$. The estimated orthogonal polynomial equation is found by substituting the estimates of $\mu, \alpha_1, \alpha_2, \dots$ into equation (2).

For the multiple equal-step input variables $X_1, X_2, \dots, X_n$, the orthogonal polynomial equation is found in a similar manner as for the single input variable. It is desired to find the degree of polynomials that adequately represents the functional relationship between the response variable and the input variables. One strategy to determine the polynomial equation is to test the significance of the terms in the sequence: linear, quadratic, cubic, and so forth. The system transfer function can be developed by including the statistically significant terms in the orthogonal polynomial model. Kuehl [8] gives an example about water uptake by barley plants to illustrate procedures to formulate the functional relationship between the amount of water uptake and two controllable factors: salinity of media and age of plant.

In parameter design, we generally select several levels for each design factor and take the levels sufficiently far apart so that a wide range can be covered by these levels. For this case, we want to develop the orthogonal polynomial models for large range of the design factors. For example,

in a circuit design problem, we may choose a resistor with a value in the range from 1 ohm to 1000 ohm. In this case, it is more reasonable to sample x values by a geometric series, such as 1, 10, 100, and 1000, rather than an arithmetic series. Although the computation becomes greatly complicated when the x values follow the geometric series, we can apply the logarithmic transformation on the x values to develop the formulas of orthogonal polynomial model for this case.

Variance Transmission Equation

Six Sigma methodology strives to improve the quality by reducing the variation of the process. By virtue of the randomness of the x_i 's, Y is a random variable via the system transfer relationship. Given some requirement of the system, one of the problems in Six Sigma process is to determine the optimal variances of input variables. Instead of the system transfer function, it is desired to find the relationship of variances between the input and output variables. Let σ_Y^2 denote the variance of the output variable Y , $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ denote the variances of the input variables $X_1, X_2, \dots, X_n$, the functional relationship of variances can be expressed as a variance transmission equation, or VTE as given below:

$$\sigma_Y^2 = h(\sigma_1^2, \dots, \sigma_n^2) + \varepsilon \quad (4)$$

where ε represents the systematic departure from the true variance of Y . VTE transfers the variances of the input variables to the variance of the response variable.

Based on the information we have, different approaches can be used to develop the variance transmission equation. We know that the functional relationship between a response Y and input variables $X_1, X_2, \dots, X_n$, can be described as a system transfer function given in equation (1). The system transfer function may not exist explicitly in an analytic form, but it is possible to

ascertain a value of Y for given values of $X_1, X_2, \dots, X_n$ by experiments or simulation. In this section, several methods for estimating the variance transmission equation are described:

- Taylor series approximation, including linear and non-linear Taylor series approximation;
- Response surface methodology;
- Taguchi's VTE via Experimental Design.

In the first two methods, the system transfer function is assumed to be known as an analytic expression. In the case where the system transfer function is not known as an analytic expression, the variance transmission equation can be developed through experimental design approach proposed by Taguchi [12]. The example on Wheatstone bridge by Taguchi [12] is used to illustrate and compare different approaches. For the purpose of comparison, we use Monte Carlo simulation with a sufficiently large number of function evaluations to estimate the variance of the response for the Wheatstone bridge system,

▪ **Taylor Series Approximation**

1. The Linear Taylor Series Approximation

If the system transfer function is known and differentiable, we can expand it as a Taylor series and drop all but the constant and the linear terms. This well known technique can be used when such a linear approximation is good enough for engineering applications. Because the system transfer function is known, we can calculate the first derivative for each x_i , and evaluate them at $x_i = \mu_i, i = 1, \dots, n$. Thus the variance transmission equation can be approximated using:

$$\sigma_Y^2 = \sum_{i=1}^n \left[\frac{\partial g(\mu_1, \mu_2, \dots, \mu_n)}{\partial x_i} \right]^2 \sigma_i^2 + O(\sigma^3) \quad (5)$$

where μ_i is the expected value of x_i , $i = 1, 2, \dots, n$, and we assume that $X_1, X_2, \dots, X_n$ are independent variables. Kapur and Lamberson [7] give an example used commonly in reliability design to analyze the error in this approximation method.

The other case is where the system transfer function is not known or it is too complicated to calculate the analytic forms for the derivatives. In order to use the linearized Taylor series expansion, we can compute numerical estimates of the derivatives in the Taylor's expansion.

For each x_i , it is possible to determine the following values by experiments [5] :

$$\begin{aligned} y_i^+ &= g(\mu_1, \dots, \mu_i + \Delta_i, \dots, \mu_n) \\ y_i^- &= g(\mu_1, \dots, \mu_i - \Delta_i, \dots, \mu_n) \end{aligned}$$

and then, we have

$$\frac{\partial g(\mu_1, \mu_2, \dots, \mu_n)}{\partial x_i} \approx \frac{y_i^+ - y_i^-}{2\Delta_i}$$

where Δ_i has a relatively small value.

2. The Non-Linear Taylor Series Approximation

When the linearized Taylor series approximation is not good enough, the extended Taylor series approximation, or the nonlinear propagation of errors are used. In order to use the extended Taylor series approximation, we need to have the system transfer function in an analytic form, with sufficient differentiability. The usual assumption is that x_i 's are statistically independent random variables.

The basic idea is to start with the Taylor series expansion for equation (1) about the means of the x_i 's up to the sixth order [5]. Then the mean and variance of Y can be calculated by algebraic manipulation. The results up to terms of order σ^4 are given below:

$$\begin{aligned}
E[Y] &= g(\mu_1, \mu_2, \dots, \mu_n) + \frac{1}{2} \sum_i \frac{\partial^2 g}{\partial x_i^2} \bigg|_{X=\mu} \sigma_i^2 \\
&\quad + \frac{1}{6} \sum_i \frac{\partial^3 g}{\partial x_i^3} \bigg|_{X=\mu} \gamma_i \sigma_i^3 + \frac{1}{24} \sum_i \frac{\partial^4 g}{\partial x_i^4} \bigg|_{X=\mu} \Gamma_i \sigma_i^4 + O(\sigma^5) \\
V[Y] &= \sum_i \left(\frac{\partial g}{\partial x_i} \bigg|_{X=\mu} \right)^2 \sigma_i^2 + \sum_i \frac{\partial g}{\partial x_i} \bigg|_{X=\mu} \frac{\partial^2 g}{\partial x_i^2} \bigg|_{X=\mu} \gamma_i \sigma_i^3 \\
&\quad + \frac{1}{3} \sum_i \frac{\partial g}{\partial x_i} \bigg|_{X=\mu} \frac{\partial^3 g}{\partial x_i^3} \bigg|_{X=\mu} \Gamma_i \sigma_i^4 + O(\sigma^5)
\end{aligned} \tag{6}$$

where $\gamma_i \sigma_i^3 = E[(x_i - \mu_i)^3] = 3^{rd}$ central moment

$\Gamma_i \sigma_i^4 = E[(x_i - \mu_i)^4] = 4^{th}$ central moment

In order to use the non-linear Taylor series expansion given by equation (6) to approximate the variance transmission equation, we need not only the first derivative but also the higher order derivatives.

▪ Response Surface Methodology

Response surface methodology (RSM) has become an important tool in product and process development, including both parameter design and tolerance design. The applications can be found in many industrial settings where several variables influence the desired outcome (e.g. minimum fraction defective or maximum yield), including the semiconductor, electronic, automotive, chemical, and pharmaceutical industries. RSM consists of techniques from mathematical optimization and statistics, and which can be used to develop new processes or improve existing ones.

Products and their manufacturing processes are influenced both by factors that are controlled by designers (called control factors or design factors) and by difficult-to-control factors such as

environmental conditions, raw material variability, and aging (called noise factors). RSM can be used for the parameter design of products or processes that may be sensitive to uncontrollable or noise factors. By developing a model containing both the noise variables and the controllable variables, a combination of controllable variable setting can be determined such that the response will be "robust" to changes in the noise variables. Sometimes a control factor and a noise factor may relate to the same variable: for instance, a nominal value and a deviation from the nominal value due to noise factors.

In order to further improve the system, the designer should set levels for the noise factors and investigate the size of their effect. Myers and Montgomery [10] describe the ideas of response surface methodology, and how to apply it to find both mean and variance models, which can be used for optimization models given later on in this paper.

The first step in RSM is to find a suitable approximation for $g(\cdot)$ in equation (1) using a lower order polynomial of the input variables. A first-order response model can be written as

$$Y = b_0 + b_1X_1 + b_2X_2 + \cdots + b_nX_n + \varepsilon$$

For a system with nonlinear behavior, a second-order response model is used as given below:

$$Y = b_0 + \sum_i b_iX_i + \sum_i b_{ii}X_i^2 + \sum_i \sum_j d_{ij}X_iX_j + \varepsilon$$

Least squares estimation is used to estimate the coefficients in the polynomials. Then a variance transmission equation is obtained by applying the variance operators to both sides of the response model, assuming that the variables are independent.

▪ Taguchi's VTE via Experimental Design

Previously, we assume that the system transfer function is available in the sense that it is either known as an analytic expression or developed through linear regression methods. If this is not

the case, then the variance transmission equation can be developed through experimental design approach. This approach is generally attributed to Taguchi [11], where a variance transmission equation is developed with the assumption of no interaction between the components. The VTE developed by Taguchi has the advantage that we can develop it by experimentation or simulation even if the analytical form of the system transfer function is not known. Also the total number of evaluations of the function is significantly less than that required by a Monte Carlo simulation.

The equation is intuitively appealing but no solid theoretical basis is available, and the interactions between the components are overlooked. By using the methodology of design of experiments, we explore the theoretical explanation of the VTE for the linear fixed effect model, regardless of the significance of the interactions between the components. Moreover, another VTE is developed for the linear random effect model. (Appendix A)

Taguchi [12] assumes that the three levels of noise factors with equal mass are set as:

$$\begin{cases} \text{Level 1:} & \mu_i - h\sigma_i \\ \text{Level 2:} & \mu_i \\ \text{Level 3:} & \mu_i + h\sigma_i \end{cases}$$

where $h = \sqrt{3/2}$. It can be shown that the above equal-mass three-point discrete distribution has the same first three moments as the normal distribution of the factor. One of the problems with using $h = \sqrt{3/2}$ is that the approximation of the higher-order moments of the distribution is sometimes poor. D'Errico and Zaino [4] propose other selections of the noise levels such as $h = \sqrt{3}$ to give a better approximation for the true distribution. Under the assumption that the factor has a probability $1/6$ to take the values of the 1st level and the 3rd level, and a probability $4/6$ to take the value of the 2nd level, this discrete distribution can match the first five moments of

the true distribution. Because one must use unequal-weighted data to compute the sum of squares and perform the analysis, the implementation of the D'Errico and Zaino's recommendation for design of experiments and the analysis seems much more complicated. For a better approximation to the true distribution of noise factors, however, the recommended $h = \sqrt{3}$ is applied to simulate the three-level noise factor for Wheatstone bridge as an improvement to Taguchi's results as given below.

▪ **Wheatstone Bridge Circuit Design**

Taguchi [12] uses the Wheatstone bridge as a case study to illustrate the procedures of parameter design and tolerance design. The Wheatstone bridge in Figure 3 is used to determine an unknown resistance Y by adjusting a known resistance so that the measured current is zero. The resistor B is adjusted until the current X registered by the galvanometer is zero, at which point the resistance value B is read and Y is calculated from the formula, $Y=BD/C$. Due to the measurement error, the current is not exactly zero, but could be a positive or negative value of about $0.2mA$. In this case the resistance is given by the following well-known approximation for the true value:

$$Y = \frac{BD}{C} - \frac{X}{C^2 E} [A(C + D) + D(B + C)] [B(C + D) + F(B + C)]$$

The noise factors in the problem are variability of the bridge components, resistors A, C, D, F, and input voltage E. This is the case where control factors and noise factors are related to the same variables. Another noise factor is the error in reading the galvanometer X. Assuming that when the galvanometer is read as zero, there may actually be a current about $0.2mA$. The inner and outer arrays are both L_{36} orthogonal arrays for the design of the experiment.

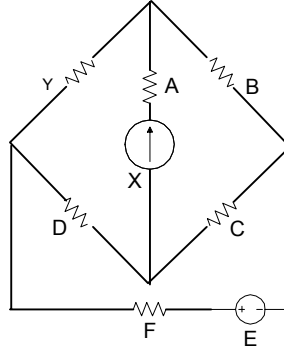


Figure 3. Wheatstone Bridge and Parameter Symbols

Taguchi did the parameter design by signal-to-noise analysis. When parameter design cannot sufficiently reduce the effect of internal and external noises, it becomes necessary to develop the variance transmission equation for the Wheatstone bridge system, and then control the variation of the major noise factors to by reducing their tolerances even though this increases the cost.

Let the nominal values or mean of control factors be the second level, the deviations due to the noise factors be the first and third level. The three levels of noise factors for the optimum combination based on parameter design are given in Table 1.

Table 1. Levels of Noise Factors for Optimum Combination

Factor	Level 1	Level 2	Level 3
A (Ω)	19.94	20	20.06
B (Ω)	9.97	10	10.03
C (Ω)	49.85	50	50.15
D (Ω)	9.97	10	10.03
E (V)	28.5	30	31.5
F (Ω)	1.994	2	2.006
X (A)	-0.0002	0	0.0002

Table 2. Comparison of Results from Different Methods

Methods	VTE	σ_Y^2
Linear Taylor	$\sigma_Y^2 = 0.04000\sigma_B^2 + 0.00160\sigma_C^2 + 0.04000\sigma_D^2 + 276.00284\sigma_X^2 + O(\sigma^3)$	7.93901E-5
Non-linear Taylor	$\sigma_Y^2 = 0.04000\sigma_B^2 + 0.00160\sigma_C^2 + 0.04000\sigma_D^2 + 276.00284\sigma_X^2 + 3.84 \times 10^{-6}\sigma_C^4 + O(\sigma^5)$	7.93910E-5
RSM (L₃₆)	$\sigma_Y^2 = 0.04004\sigma_B^2 + 0.00162\sigma_C^2 + 0.04036\sigma_D^2 + 300.37396\sigma_X^2 + 1.42 \times 10^{-8}$	8.04528E-5
RSM (2187)	$\sigma_Y^2 = 0.04000\sigma_B^2 + 0.00160\sigma_C^2 + 0.04000\sigma_D^2 + 299.59875\sigma_X^2 + 1.00 \times 10^{-8}$	8.00003E-5
IPV RSM	$\sigma_Y^2 = 0.04000\sigma_B^2 + 0.00160\sigma_C^2 + 0.04000\sigma_D^2 + 299.60130\sigma_X^2 + 1.43 \times 10^{-8}$	8.00051E-5
Taguchi (L₃₆)	$\sigma_Y^2 = 0.04118\sigma_B^2 + 0.00166\sigma_C^2 + 0.04150\sigma_D^2 + 308.93935\sigma_X^2 + 1.42 \times 10^{-8}$	8.27494E-5
Taguchi (2187)	$\sigma_Y^2 = 0.04002\sigma_B^2 + 0.00160\sigma_C^2 + 0.04002\sigma_D^2 + 299.73565\sigma_X^2 + 1.00 \times 10^{-8}$	8.00408E-5
IPV Taguchi	$\sigma_Y^2 = 0.04000\sigma_B^2 + 0.00160\sigma_C^2 + 0.04000\sigma_D^2 + 299.76800\sigma_X^2 + 1.43 \times 10^{-8}$	8.00095E-5
Monte Carlo	1,000,000 observations	7.99860E-5

Note:

- The calculation of σ_Y^2 is for $\sigma_B=0.02449$, $\sigma_C=0.12247$, $\sigma_D=0.02449$, $\sigma_X=0.00016$;
- RSM (2187) is the response surface method applied on the same data set as the Taguchi's VTE (2187)
- Improved (IPV) RSM is the response surface method applied on the same data set as the Improved (IPV) Taguchi's VTE.

The results from all of the methods discussed before are compared in Table 2. RSM (L₃₆) and Taguchi (L₃₆) have the same L₃₆ orthogonal array design layout for comparison purpose in Table 2. Improved RSM and Improved Taguchi use the complete design with $N=3^7=2187$ data points for the un-equal mass three-level noise factors. For the purpose of comparison, we also perform the complete design with 2187 data points for the equal-mass three-level noise factors, which are denoted as RSM (2187) and Taguchi (2187) in Table 2. Without considering the different design layouts, it seems that the improved method gives better approximation of variance. We can see that the improved Taguchi's VTE does not differ much from the original one in their ability to approximate the variance of the response. Because the improved Taguchi's method requires the

complete evaluation at all combinations of levels, it is costly in terms of time and resources. If the high cost of the complete design is a concern, the original Taguchi's equal-mass three-level method using orthogonal array is preferred. If the complete evaluation can be accomplished by simulation without much difficulty, the improved Taguchi's method should be applied to ensure the high accuracy.

Economic Optimization and Quality Improvement

The ultimate objective of Six Sigma strategy is to minimize the total cost to both the producer and consumer, or the whole system. The cost to the consumer is related to the expected quality loss of the output variable, and it is caused by the deviation from the target value. The cost to the producer is associated with changing probability distributions of input variables. If the system transfer function and the variance transmission equation are available, and the cost functions for different grades of input factors are given, the general optimization model to reflect the optimization strategy is given in Figure 4.

• General Optimization Problem

We usually consider the first two moments of the probability distributions of input variables, and then the optimization models will focus on the mean and variance values. Therefore, the expected quality loss to the consumer consists of two parts: the bias of the process and the variance of the process. The strategy to reduce bias is to find adjustment factors that do not affect variance, and thus are used to bring the mean closer to the target value. Design of experiments can be used to find these adjustment factors. It will incur certain cost to the producer. In order to reduce the variance of Y , the designer should reduce the variances of the input variables and that

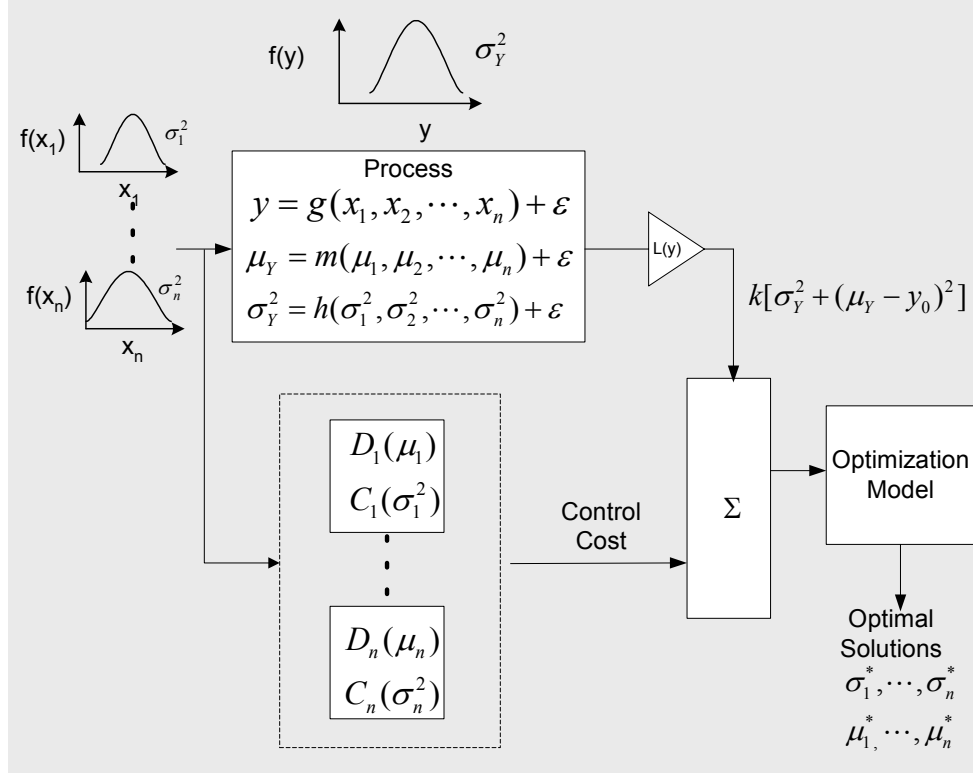


Figure 4. General Optimization Model for the System

will also increase cost. The problem is to balance the reduced expected quality loss with the increased cost for the reduction of the bias and variances of the input variables. Typically, the variance control cost for the i^{th} input variable X_i is denoted by $C_i(\sigma_i^2)$, and the mean control cost for the i^{th} input variable X_i is denoted by $D_i(\mu_i)$. By focusing on the first two moments of the probability distributions of $X_1, X_2, \dots, X_n$, the general optimization model is formulated as

$$\text{Minimize } TC = \sum_{i=1}^n C_i(\sigma_i^2) + \sum_{i=1}^n D_i(\mu_i) + k[\sigma_Y^2 + (\mu_Y - y_0)^2] \quad (7)$$

$$\text{subject to } \mu_Y \approx m(\mu_1, \mu_2, \dots, \mu_n)$$

$$\sigma_Y^2 \approx h(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2)$$

In this objective function, the first two terms $\sum_{i=1}^n C_i(\sigma_i^2)$ and $\sum_{i=1}^n D_i(\mu_i)$, are the control costs on the variances and means of input variables, or the cost to the producer; and the last term $k[\sigma_Y^2 + (\mu_Y - y_0)^2]$, is the expected quality loss to the customer, where k is a constant in the quality loss function. The first constraint $\mu_Y \approx m(\mu_1, \mu_2, \dots, \mu_n)$, is the model for the mean of the system, which can be obtained through the system transfer function. The second constraint $\sigma_Y^2 \approx h(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2)$, is the variance transmission equation. An example of this general optimization problem can be an electronic circuit example given by Jeang [6]. A research problem for future is to solve this optimization problem which considers together both the mean and the variance.

• Tolerance Design Problem

If we assume that the bias reduction has been accomplished, the general optimization problem given by Equation (7) can be simplified as a tolerance design problem, which is given below:

$$\begin{aligned} \text{Minimize} \quad & TC = \sum_{i=1}^n C_i(\sigma_i^2) + k\sigma_Y^2 \\ \text{subject to} \quad & \sigma_Y^2 \approx h(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) \end{aligned} \quad (8)$$

The objective of the tolerance design is to determine the tolerances (which are related to variances) of the input variables to minimize the total cost, which consists of the expected quality loss due to variation $k\sigma_Y^2$, and the control cost on the tolerances of the input variables

$\sum_{i=1}^n C_i(\sigma_i^2)$. Typically, $C_i(\sigma_i^2)$ is a non-increasing function of each σ_i^2 .

For this tolerance design problem, a RLC circuit example is given by Chen [3] to minimize the total cost to both the manufacturer and the consumer. Taguchi's method is used to construct the

variance transmission equation as the constraint in Chen's example. Bare, Kapur and Zabinsky [1] propose another optimization model to minimize the total variance control cost by finding the optimum standard deviations of input variables. Taylor's series expansion is used to develop the variance transmission equation in their model.

• Dual Problem of Tolerance Design Problem

In addition, given the constraint on the control cost of the tolerances, the dual problem of the tolerance design problem can be developed to minimize the variance of response as given below:

$$\begin{aligned} \text{Minimize} \quad & \sigma_Y^2 \approx h(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) \\ \text{subject to} \quad & \sum_{i=1}^n C_i(\sigma_i^2) \leq C^* \end{aligned} \quad (9)$$

where C^* is the maximum allowable cost to the producer, or control cost on the tolerances of input variables.

Conclusions

Six Sigma and other continuous improvement strategies are extremely important tools for the global competition. The research ideas presented in this paper are important in terms of the basic research and their application for analysis and improvement phases of the (D)MAIC(T) process. It will contribute towards the design of many products and processes and also improve the quality and productivity in any organization.

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Appendix A. Theoretical Explanation and Improvement of Taguchi's VTE

For two factors A and B, the general model for a completely randomized design with n observations per treatment combination is of the form

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad (\text{A.1})$$

with $i = 1, 2, \dots, a$; $j = 1, 2, \dots, b$; $k = 1, 2, \dots, n$.

where α_i is the main effect of factor A at the i^{th} level; β_j is the main effect of factor B at the j^{th} level; $(\alpha\beta)_{ij}$ is the interaction effect between factor A at the i^{th} level and factor B at the j^{th} level; μ is a grand mean with constant value and ε_{ijk} are random experimental error with normal distribution $\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$.

If both sides of the above equation are squared and summed over all observations, the left-hand side is the total sum of squares, denoted by SS_T . SS_T is partitioned into three components represented by the effects on the right-hand side of the above equation:

$$SS_T = SS_A + SS_B + SS_{AB} + SS_e \quad (\text{A.2})$$

A.1. VTE for Linear Fixed Effect Model

If both A and B are at fixed levels, the model is a fixed effect model. The following ANOVA table A.1 summarizes the data for the fixed effect model, including the expected mean square (EMS) data for each effect.

Table A.1. ANOVA Table for Fixed Effect Model

Source	df	SS	MS	EMS	SS'	$\hat{\rho}$
A	a-1	SS _A	MS _A	$\sigma_{\varepsilon}^2 + nb \frac{\sum_i \alpha_i^2}{a-1}$	SS _A -(a-1) MS _e	$\frac{SS'_A}{SS_T}$
B	b-1	SS _B	MS _B	$\sigma_{\varepsilon}^2 + na \frac{\sum_j \beta_j^2}{b-1}$	SS _B -(b-1) MS _e	$\frac{SS'_B}{SS_T}$
AB	(a-1)(b-1)	SS _{AB}	MS _{AB}	$\sigma_{\varepsilon}^2 + n \frac{\sum_i \sum_j (\alpha\beta)_{ij}^2}{(a-1)(b-1)}$	SS _{AB} -(a-1)(b-1) MS _e	$\frac{SS'_{AB}}{SS_T}$
error	ab(n-1)	SS _e	MS _e	σ_{ε}^2		$\frac{MS_e}{MS_T}$
Total	Abn-1	SS _T	MS _T	σ_y^2		1.00

By taking the expected value on both sides of equation (A.2), we have

$$\begin{aligned}
 (abn-1)EMS_T &= (a-1)EMS_A + (b-1)EMS_B + (a-1)(b-1)EMS_{AB} + (abn-ab)\sigma_{\varepsilon}^2 \\
 &= bn \sum_i \alpha_i^2 + an \sum_j \beta_j^2 + n \sum_i \sum_j (\alpha\beta)_{ij}^2 + (abn-1)\sigma_{\varepsilon}^2
 \end{aligned}$$

so that we have

$$EMS_T = \frac{bn \sum_i \alpha_i^2 + an \sum_j \beta_j^2 + n \sum_i \sum_j (\alpha\beta)_{ij}^2}{abn-1} + \sigma_{\varepsilon}^2 \quad (A.3)$$

For fixed effect model, if the null hypotheses are that treatment and interaction effects are zero, then EMS_T estimates the experiment error, σ_{ε}^2 . However, if there are treatment effects or interaction present, then the corresponding EMS_T can estimate the total variance of the system, σ_y^2 , which includes not only the experiment error, but also the variance caused by the variability of treatment effects and interaction.

If we can find the relationships between $\sum \alpha_i^2 \sim \sigma_A^2$, $\sum \beta_j^2 \sim \sigma_B^2$, $\sum \sum (\alpha\beta)_{ij}^2 \sim \sigma_A^2$ and σ_B^2 , equation (A.3) can be further developed into an ideal variance transmission equation. Using this

idea, VTE will be developed later for the linear effect model. Before that, two important concepts related to the development of VTE will be introduced: Contribution Ratio ρ and Pure Sum of Squares SS' .

Based on equation (A.3), the contribution ratios are defined as below:

$$\rho_A = \frac{nb \sum_i \alpha_i^2}{(abn-1)EMS_T}; \rho_B = \frac{na \sum_j \beta_j^2}{(abn-1)EMS_T}; \rho_{AB} = \frac{n \sum \sum (\alpha\beta)_{ij}^2}{(abn-1)EMS_T}; \rho_e = \frac{\sigma_e^2}{EMS_T}$$

so that all the contribution ratios will add up to 1. In order to use the values in the ANOVA table to estimate the contribution ratios, we introduce the pure sum of squares to estimate the numerators of the contribution ratios, which are denoted by SS' as in Table A.1. Then the contribution ratio can be estimated based on the values in ANOVA table as

$$\begin{aligned} \hat{\rho}_A &= \frac{SS'_A}{SS_T} = \frac{SS_A - (a-1)MS_e}{SS_T} \\ \hat{\rho}_B &= \frac{SS'_B}{SS_T} = \frac{SS_B - (b-1)MS_e}{SS_T} \\ \hat{\rho}_{AB} &= \frac{SS'_{AB}}{SS_T} = \frac{SS_{AB} - (a-1)(b-1)MS_e}{SS_T} \\ \hat{\rho}_e &= \frac{MS_e}{MS_T} \end{aligned}$$

The pure sum of squares and the estimated contribution ratios are listed in the last two columns of Table A.1.

For the linear effect model in equation (A.1), the treatment effects and interaction effect have the values

$$\begin{aligned} \alpha_i &= \bar{y}_{i..} - \mu \\ \beta_j &= \bar{y}_{.j.} - \mu \\ (\alpha\beta)_{ij} &= \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \mu \end{aligned}$$

We can use the regression model to link the response y with the factors A and B. For the two-factor factorial experiment, the regression model for linear effect model could be written as

$$Y_{ijk} = \lambda_0 + \lambda_1 A_i + \lambda_2 B_j + \lambda_{12} A_i B_j + \varepsilon_{ijk}$$

when A and B are linear effect, and we can prove that

$$\begin{aligned}\alpha_i &= (\lambda_1 + \lambda_{12} \mu_B)(A_i - \mu_A) = k_A(A_i - \mu_A) \\ \beta_j &= (\lambda_2 + \lambda_{12} \mu_A)(B_j - \mu_B) = k_B(B_j - \mu_B) \\ (\alpha\beta)_{ij} &= \lambda_{12}(A_i - \mu_A)(B_j - \mu_B)\end{aligned}\tag{A.4}$$

where k_A and k_B are constants. Then we have

$$\begin{aligned}E\left[\frac{\sum_i \alpha_i^2}{a-1}\right] &= E\left[\frac{\sum (k_A(A_i - \mu_A))^2}{a-1}\right] = k_A^2 E[s_A^2] \\ E\left[\frac{\sum_j \beta_j^2}{b-1}\right] &= E\left[\frac{\sum (k_B(B_j - \mu_B))^2}{b-1}\right] = k_B^2 E[s_B^2]\end{aligned}$$

where s_A^2 and s_B^2 are sample variances of factors A and B. We can prove that s_A^2 and s_B^2 are unbiased estimator of σ_A^2 and σ_B^2 , respectively. Also we can show below that

$$E\left[\frac{\sum_i \sum_j (\alpha\beta)_{ij}^2}{(a-1)(b-1)}\right] = \lambda_{12}^2 \sigma_A^2 \sigma_B^2.$$

For any two random variables W and V, we have [7]

$$\begin{aligned}E[W^2 V^2] &= (E[WV])^2 + V[WV] \\ V[WV] &= \sigma_W^2 \sigma_V^2 + \sigma_V^2 \mu_W^2 + \sigma_W^2 \mu_V^2\end{aligned}$$

If W and V are independent, then $E[WV] = E[W]E[V]$. Specifically, when $E[W] = E[V] = 0$, we have $E[W^2 V^2] = \sigma_W^2 \sigma_V^2$. Because we assume that A and B are independent variables, we have

that $A - \mu_A$ and $B - \mu_B$ are also independent and $E[A - \mu_A] = E[B - \mu_B] = 0$. It follows that

$E[(A - \mu_A)^2(B - \mu_B)^2] = \sigma_A^2 \sigma_B^2$. If we take a sample of size a from A and a sample of size b

from B , we can prove that $\frac{\sum_i \sum_j ((A_i - \mu_A)(B_j - \mu_B))^2}{(a-1)(b-1)}$ is an unbiased estimator of

$E[(A - \mu_A)^2(B - \mu_B)^2]$. Thus we have

$$E\left[\frac{\sum_i \sum_j (\alpha\beta)_{ij}^2}{(a-1)(b-1)}\right] = E\left[\frac{\sum_i \sum_j (\lambda_{12}(A_i - \mu_A)(B_j - \mu_B))^2}{(a-1)(b-1)}\right] = \lambda_{12}^2 \sigma_A^2 \sigma_B^2$$

and we also have

$$\begin{aligned}\rho_A &= \frac{bn(a-1)k_A^2 \sigma_A^2}{(abn-1)EMS_T} = \frac{c_1 \sigma_A^2}{EMS_T} \\ \rho_B &= \frac{an(b-1)k_B^2 \sigma_B^2}{(abn-1)EMS_T} = \frac{c_2 \sigma_B^2}{EMS_T} \\ \rho_{AB} &= \frac{n(a-1)(b-1)\lambda_{12}^2 \sigma_A^2 \sigma_B^2}{(abn-1)EMS_T} = \frac{c_3 \sigma_A^2 \sigma_B^2}{EMS_T} \\ \rho_e &= \frac{\sigma_\varepsilon^2}{EMS_T}\end{aligned}$$

where c_1, c_2 and c_3 constants.

Then equation (A.3) can be modified as

$$\sigma_Y^2 = c_1 \sigma_A^2 + c_2 \sigma_B^2 + c_3 \sigma_A^2 \sigma_B^2 + \sigma_\varepsilon^2 \quad (\text{A.5})$$

Given another set of components with new variances $\sigma_A'^2, \sigma_B'^2$, we can get the new $\sigma_Y'^2$ from

equation (A.5) without again performing the experiments or simulations as given below:

$$\begin{aligned}\sigma_Y'^2 &= \frac{\rho_A EMS_T}{\sigma_A^2} \sigma_A'^2 + \frac{\rho_B EMS_T}{\sigma_B^2} \sigma_B'^2 + \frac{\rho_{AB} EMS_T}{\sigma_A^2 \sigma_B^2} \sigma_A'^2 \sigma_B'^2 + \sigma_\varepsilon^2 \\ &= \sigma_Y^2 \left[\rho_A \frac{\sigma_A'^2}{\sigma_A^2} + \rho_B \frac{\sigma_B'^2}{\sigma_B^2} + \rho_{AB} \frac{\sigma_A'^2 \sigma_B'^2}{\sigma_A^2 \sigma_B^2} + \rho_\varepsilon \right]\end{aligned}$$

where the contribution ratios can be estimated by the last column of Table 1.

A.2. VTE for Linear Random Effect Model

If both A and B are at random levels, the model is a random effect model. The following ANOVA table summarizes the data for a random model, including the expected mean square (EMS) data for each effect.

Table A.2. ANOVA Table for Random Effect Model

Source	df	SS	MS	EMS	Est. Var. Component.
A	a-1	SS _A	MS _A	$\sigma_\varepsilon^2 + n\sigma_{\alpha\beta}^2 + nb\sigma_\alpha^2$	$\hat{\sigma}_\alpha^2 = \frac{MS_A - MS_{AB}}{nb}$
B	b-1	SS _B	MS _B	$\sigma_\varepsilon^2 + n\sigma_{\alpha\beta}^2 + na\sigma_\beta^2$	$\hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{na}$
AB	(a-1)(b-1)	SS _{AB}	MS _{AB}	$\sigma_\varepsilon^2 + n\sigma_{\alpha\beta}^2$	$\hat{\sigma}_{\alpha\beta}^2 = \frac{MS_{AB} - MS_e}{n}$
error	ab(n-1)	SS _e	MS _e	σ_ε^2	
Total	Abn-1	SS _T	MS _T	σ_y^2	

In a random effect model, the experimenter is interested in estimating the variance components.

Given the effect model in equation (A.1), the variance of system can be decomposed as

$$\sigma_Y^2 = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \sigma_\varepsilon^2$$

where σ_α^2 and σ_β^2 are the variances due to the main effects of A and B, $\sigma_{\alpha\beta}^2$ is the variance due to the interaction effect of AB, and σ_ε^2 is the variance due to random error. In Table A.2, these variance components are estimated and given in the last column.

If only the linear effects of factors A and B are significant, we can easily derive from equation (A.4) that

$$\begin{aligned}\sigma_{\alpha}^2 &= (k_A \sigma_A)^2 \\ \sigma_{\beta}^2 &= (k_B \sigma_B)^2 \\ \sigma_{\alpha\beta}^2 &= (\lambda_{12} \sigma_A \sigma_B)^2\end{aligned}$$

Then the variance transmission equation for linear random effect model can be developed as

$$\sigma_Y^2 = k_A^2 \sigma_A^2 + k_B^2 \sigma_B^2 + \lambda_{12}^2 \sigma_A^2 \sigma_B^2 + \sigma_{\varepsilon}^2 \quad (\text{A.6})$$

Given another set of components with new variances $\sigma_A'^2, \sigma_B'^2$, we can get the new $\sigma_Y'^2$ from equation (A.6) without performing the experiment or simulation again as given below:

$$\sigma_Y'^2 = \frac{\hat{\sigma}_{\alpha}^2}{\sigma_A^2} \sigma_A'^2 + \frac{\hat{\sigma}_{\beta}^2}{\sigma_B^2} \sigma_B'^2 + \frac{\hat{\sigma}_{\alpha\beta}^2}{\sigma_A^2 \sigma_B^2} \sigma_A'^2 \sigma_B'^2 + \sigma_{\varepsilon}^2 \quad (\text{A.7})$$

where the estimated variance components can be obtained from the last column of Table A.2.