Appendix. Proofs of all the Lemmas and Propositions.

***Lemma* 1.** *Let the level of knowledge at time t be K. Then the expected value of the incremental cash flows due to process improvement at time t,* $1 \le t \le N$*, is*

$$V(t,K) = I(t)\exp(\overline{C}(K) - \beta_C(K))\Pi[r,t]$$

where

$$\Pi[r,t] = \sum_{s=t+1}^{N} e^{-(s-t)r} + \sum_{s=N+1}^{\infty} \vartheta^{s-N} e^{-(s-t)r} \qquad \text{(A1)}$$

is the value of a perpetual consul bond, where the payment on this bond, after N, depreciates at a constant rate 1 - ϑ*.*

Proof of Lemma 1. The value of the incremental cash flows related to process improvement at time *t* is

$$V(t,K) = E_t\left[\sum_{s=t+1}^{N} \frac{z(s)}{z(t)} C_t(s,K) + \sum_{s=N+1}^{\infty} \frac{z(s)}{z(t)} \chi(s) C_t(s,K)\right] \qquad \text{(A2)}$$

where $1 \le t \le N$. For $s > N$,

$$E_t\left[\frac{z(s)}{z(t)} \chi(s) C_t(s,K)\right]$$

$$= \vartheta^{s-N} E_t\left[\frac{z(s-1)}{z(t)} E_{s-1}\left\{\frac{z(s)}{z(s-1)} C_t(s,K)\right\}\right]$$

$$= \vartheta^{s-N} I(t)\exp(\overline{C}(K) - \beta_C(K))\exp(-r) E_t\left[\frac{z(s-1)}{z(t)}\right]$$

$$= \vartheta^{s-N} I(t)\exp(\overline{C}(K) - \beta_C(K))\exp(-(s-t)r)$$

where the first equality follows from the law of iterated expectations (see Billingsley 1986, page 470) and the fact: $\chi(s)$ is independent of all other random variables in the model and

$E[\chi(s)=1/\chi(N)=1]=\vartheta^{s-N}$. The second equality follows from the lognormality of the cash flows and the pricing kernel, which yields

$$E_{s-1}\left[\frac{z(s)}{z(s-1)}C_t(s,K)\right]=\exp[-r]I(t)\exp[\overline{C}(K)-\beta_C(K)].$$

The third equality follows from the law of iterated expectations (see Billingsley 1986, page 470) and the lognormality of the pricing kernel. Similarly, for $s \le N$,

$$E_t\left[\frac{z(s)}{z(t)}C_t(s,K)\right]= I(t)\exp(\overline{C}(K)-\beta_C(K))\exp(-(s-t)r)$$

Substituting each term in equation (A2) yields the result. □

Before we go ahead to prove Lemma 2, we first rewrite the definition of $M(t,K)$ as follows

$$M(t,K)=\pi(K)\exp(\overline{C}(K)-\beta_C(K))\Pi[r,t] \qquad \text{(A3)}$$

Lemma 2: Given the definition of $M(t,K)$ in (A3), we have the following properties:

(a) $M(t,K+1)>M(t,K)$, $1\le t\le N$

(b) $M(t,K)>M(t+1,K)$, $1\le t<N$

(c) $M(t,K+1)-M(t+1,K+1)>M(t,K)-M(t+1,K)$, $1\le t<N$

Proof of Lemma 2. Given the definition of $M(t,K)$ in (A3), it is straightforward to show that

$$M(t,K+1)>M(t,K)$$

since $\pi(K)$ and $\overline{C}(K)$ are increasing with K and $\beta_C(K)$ decreasing with K.

To show that $M(t,K)>M(t+1,K)$, we show that $\Pi(t,r)>\Pi(t+1,r)$.

From equation (A1), we have

$$\Pi(t,r)=e^{-r}+e^{-2r}+\cdots+e^{-(N-t)r}+\vartheta e^{-(N+1-t)r}+\vartheta^2 e^{-(N+2-t)r}+\cdots$$

$$\Pi(t+1,r)=e^{-r}+e^{-2r}+\cdots+e^{-(N-1-t)r}+\vartheta e^{-(N-t)r}+\vartheta^2 e^{-(N+1-t)r}+\vartheta^3 e^{-(N+2-t)r}+\cdots$$

Thus,

$$\Pi(t,r)-\Pi(t+1,r)=(1-\vartheta)(\Pi(t,r)-e^{-r}-e^{-2r}-\cdots-e^{-(N-1-t)r})>0\,.$$

Thus $M(t,K)-M(t+1,K)=\pi(K)\exp(\overline{C}(K)-\beta_C(K))\left(\Pi(t,r)-\Pi(t+1,r)\right)>0$

Again since $\pi(K)$ and $\overline{C}(K)$ are increasing with K and $\beta_C(K)$ decreasing with K, we have

$M(t,K+1)-M(t+1,K+1)>M(t,K)-M(t+1,K)$, $1\le t<N$. □

Proposition* 1*: Optimal investment decisions.

At any time t, $1\le t\le N$ ***,*** *with a knowledge level of K, if* $M(t,K)+\Omega(t,K)>1$*, then the firm should go ahead to invest in process improvement; otherwise, the firm should not invest in process improvement, where* $M(t,K)$ *has been defined earlier and* $\Omega(t,K)$ *is calculated as:*

$$\Omega(t,K)=\sum_{j=1}^{N-t}W(t+j,K)=W(t+1,K)+\Omega(t+1,K) \qquad \text{(A4)}$$

$$W(t,K)=\max\left[M(t,K+1)+\Omega(t,K+1)-1,\,0\right]-\max\left[M(t,K)+\Omega(t,K)-1,\,0\right] \qquad \text{(A5)}$$

$$\Omega(N,K+j)=0,\quad j=0,1,\,\cdots,\,N-t+2\,. \qquad \text{(A6)}$$

Proof of Proposition 1. Proposition 1 is a special case of Proposition 5, so its proof is omitted. □

<u>Derivation of the condition for the firm not to invest at or after time period *N*.</u>

In order to make sure that the firm has no economic incentive to invest at or after N, we need to make sure that $M(N,K)<1$, since the value in the change of options due to the increase of the knowledge level is zero if the value of the cash flows for investment in the process improvement is smaller than the cost and the firm thus doesn't invest at or after period *N*.
Since

$$M(N,K)=\pi(K)\exp(\overline{C}(K)-\beta_C(K))\Pi[r,N]\text{ , and }\Pi[r,N]=\sum_{t=1}^{\infty}\left(\frac{\vartheta}{e^r}\right)^t$$

We have

$$M(N,K)=\pi(K)\exp(\overline{C}(K)-\beta_C(K))\Pi[r,N]=\pi(K)\exp(\overline{C}(K)-\beta_C(K))\vartheta/(e^r-\vartheta)$$

To make $M(N,K)<1$, then $\vartheta<\dfrac{e^r}{\partial(K)+1}$, $\partial(K)=\pi(K)\exp(\overline{C}(K)-\beta_C(K))$.

$\partial(K)$ is increasing at K. The maximum knowledge level at N is $K_0 + N$ -1.

Thus, if $\vartheta<\dfrac{e^r}{\partial(K_0+N-1)+1}$, the firm will not invest at or after time N.

Lemma 3. *Let K be the knowledge level at time t and* $\Omega(t,K)$ *be defined in the above. Then,*

$\Omega(t,K)\geq 0$, $1\leq t\leq N$.

Proof of Lemma 3. We prove by induction. The conclusion is easy to prove when t=N, since $\Omega(N,K)=0$ according to the definition in (A6). When $t=N-1$, from equations (A4) to (A6), $\Omega(N-1,K)=Max[M(N,K+1)-1,0]-Max[M(N,K)-1,0]\geq 0$, since $M(N,K+1)>M(N,K)$.

Suppose the conclusion is valid when $t=i+1$. That is, $\Omega(i+1,K)\geq 0$. Since K can be any finite non-negative integer, clearly $\Omega(i+1,K+1)\geq 0$, which will be used in the following.

When $t=i$, according to equation (A4), $\Omega(i,K)=W(i+1,K)+\Omega(i+1,K)$,

where $W(i+1,K)=Max[M(i+1,K+1)+\Omega(i+1,K+1)-1,0]-Max[M(i+1,K)+\Omega(i+1,K)-1,0]$

We have the following four cases*:*

1: Both options are exercised:

$W(i+1,K)=M(i+1,K+1)+\Omega(i+1,K+1)-M(i+1,K)-\Omega(i+1,K)$.

So, $\Omega(i,K)=M(i+1,K+1)+\Omega(i+1,K+1)-M(i+1,K)>0$, since $M(i+1,K+1)-M(i+1,K)>0$ and $\Omega(i+1,K+1)\geq 0$

2: Only the option with knowledge K+1 is exercised. Then

$W(i+1,K)=M(i+1,K+1)+\Omega(i+1,K+1)-1\geq 0$.

$\Omega(i,K) = W(i+1,K) + \Omega(i+1,K) \geq 0$, since $\Omega(i+1,K) \geq 0$.

3: Only the option with knowledge K is exercised. Then

$W(i+1,K) = 1 - M(i+1,K) - \Omega(i+1,K)$.

$\Omega(i,K) = W(i+1,K) + \Omega(i+1,K) = 1 - M(i+1,K) > 0$. This is because the option with knowledge *K+1* is not exercised, *that is,* $1 - M(i+1,K+1) - \Omega(i+1,K+1) > 0$.

Thus, $1 - M(i+1,K) > 1 - M(i+1,K+1) \geq 1 - M(i+1,K+1) - \Omega(i+1,K+1) > 0$.

4: No option is exercised.

$W(i+1,K) = 0$.

$\Omega(i,K) = W(i+1,K) + \Omega(i+1,K) = \Omega(i+1,K) \geq 0$.

Thus $\Omega(i,K) \geq 0$. □

Proposition 2 . *Let K be the knowledge level at time t and* $\Omega(t,K)$ *be defined in the above. Then,*

$M(t,K+1) + \Omega(t,K+1) > M(t,K) + \Omega(t,K)$, $1 \leq t \leq N$.

Proof of Proposition 2. *We prove by induction. When* $t = N$, *we have*

$M(N,K+1) + \Omega(N,K+1) > M(N,K) + \Omega(N,K)$,

since $\Omega(N,K+1) = \Omega(N,K) = 0$ and $M(N,K+1) > M(N,K)$ from part (a) of Lemma 2.

When $t = i+1$, suppose $M(i+1,K+1) + \Omega(i+1,K+1) > M(i+1,K) + \Omega(i+1,K)$. Since K can be any finite non-negative integer, $M(i+1,K+2) + \Omega(i+1,K+2) > M(i+1,K+1) + \Omega(i+1,K+1)$, *which* will be used in the following.

Then, when $t = i$, we have

$$M(i,K+1) + \Omega(i,K+1) = M(i,K+1) + Max[M(i+1,K+2) + \Omega(i+1,K+2) - 1, 0] - Max[M(i+1,K+1) + \Omega(i+1,K+1) - 1, 0] + \Omega(i+1,K+1)$$

$$M(i,K) + \Omega(i,K) = M(i,K) + Max[M(i+1,K+1) + \Omega(i+1,K+1) - 1, 0] - Max[M(i+1,K) + \Omega(i+1,K) - 1, 0] + \Omega(i+1,K)$$

Since $M(i+1, K+2)+\Omega(i+1, K+2) > M(i+1, K+1)+\Omega(i+1, K+1) > M(i+1, K)+\Omega(i+1, K)$

we have the following four cases:

Case 1: $M(i+1, K+2)+\Omega(i+1, K+2) < 1$.

$M(i, K+1)+\Omega(i, K+1) = M(i, K+1)+\Omega(i+1, K+1)$ *and* $M(i, K)+\Omega(i, K) = M(i, K)+\Omega(i+1, K)$.

Since

$M(i, K+1)+\Omega(i+1, K+1) = M(i, K+1) - M(i+1, K+1) + M(i+1, K+1)+\Omega(i+1, K+1)$ *and*

$M(i, K)+\Omega(i+1, K) = M(i, K) - M(i+1, K) + M(i+1, K)+\Omega(i+1, K)$,

using the result in part (*c*) *of Lemma 2 and* $M(i+1, K+1)+\Omega(i+1, K+1) > M(i+1, K)+\Omega(i+1, K)$,

we then have

$M(i, K+1)+\Omega(i, K+1) > M(i, K)+\Omega(i, K)$.

Case 2: $M(i+1, K+2)+\Omega(i+1, K+2) \geq 1$ *and* $M(i+1, K+1)+\Omega(i+1, K+1) < 1$

$M(i, K+1)+\Omega(i, K+1) = M(i, K+1)+M(i+1, K+2)+\Omega(i+1, K+2) - 1 + \Omega(i+1, K+1)$.

$M(i, K)+\Omega(i, K) = M(i, K)+\Omega(i+1, K)$

Using $M(i, K+1)+\Omega(i+1, K+1) > M(i, K)+\Omega(i+1, K)$ *in case 1 and*

$M(i+1, K+2)+\Omega(i+1, K+2) - 1 \geq 0$, *we have*

$M(i, K+1)+\Omega(i, K+1) > M(i, K)+\Omega(i, K)$.

Case 3: $M(i+1, K+1)+\Omega(i+1, K+1) \geq 1$ *and* $M(i+1, K)+\Omega(i+1, K) < 1$

$M(i, K+1)+\Omega(i, K+1) = M(i, K+1)+M(i+1, K+2)+\Omega(i+1, K+2) - M(i+1, K+1)$

$$\begin{aligned} M(i, K)+\Omega(i, K) &= M(i, K)+M(i+1, K+1)+\Omega(i+1, K+1)+\Omega(i+1, K) - 1 \\ &< M(i, K)+M(i+1, K+1)+\Omega(i+1, K+1)+\Omega(i+1, K) - M(i+1, K) - \Omega(i+1, K) \\ &= M(i, K)+M(i+1, K+1)+\Omega(i+1, K+1) - M(i+1, K) \end{aligned}$$

Since $M(i, K+1) - M(i+1, K+1) > M(i, K) - M(i+1, K)$ *from part* (*c*) *of Lemma 2 and*

$M(i+1, K+2)+\Omega(i+1, K+2) > M(i+1, K+1)+\Omega(i+1, K+1)$

we have $M(i,K+1)+\Omega(i,K+1) > M(i,K)+\Omega(i,K)$

Case 4: $M(i+1,K)+\Omega(i+1,K) \geq 1$

$M(i,K+1)+\Omega(i,K+1) = M(i,K+1)+M(i+1,K+2)+\Omega(i+1,K+2)-M(i+1,K+1)$

$M(i,K)+\Omega(i,K) = M(i,K)+M(i+1,K+1)+\Omega(i+1,K+1)-M(i+1,K)$

Since $M(i,K+1)-M(i+1,K+1) > M(i,K)-M(i+1,K)$ *from part* (*c*) *of Lemma 2 and*

$M(i+1,K+2)+\Omega(i+1,K+2) > M(i+1,K+1)+\Omega(i+1,K+1)$, *we have*

$M(i,K+1)+\Omega(i,K+1) > M(i,K)+\Omega(i,K)$

Thus

$M(t,K+1)+\Omega(t,K+1) > M(t,K)+\Omega(t,K)$, $1 \leq t \leq N$.□

Lemma 4. *Let K be the knowledge level at time t. Then* $\Omega(t,K) \geq \Omega(t+1,K)$

Proof of Lemma 4. *First, we show that* $W(t,K) \geq 0$.

$W(t,K) = Max.[M(t,K+1)+\Omega(t,K+1)-1, 0] - Max.[M(t,K)+\Omega(t,K)-1, 0]$

Since $M(t,K+1)+\Omega(t,K+1) > M(t,K)+\Omega(t,K)$ from Proposition 2, *we have* $W(t,K) \geq 0$.

Since $W(t+1,K) \geq 0$ *and* $\Omega(t,K) = W(t+1,K)+\Omega(t+1,K)$ from equation (A4), *we have*

$\Omega(t,K) \geq \Omega(t+1,K)$.□

Proposition 3. *Let K be the knowledge level at time t. Then*

$M(t,K)+\Omega(t,K) > M(t+1,K)+\Omega(t+1,K)$, $1 \leq t < N$.

So, if the firm doesn't invest at time t, the firm will not invest in the remaining periods.

Proof of Proposition 3. *The inequality follows directly from part (b) of Lemma 2 and Lemma 4.*

Now, if the firm does not invest at time t, then $M(t,K)+\Omega(t,K) < 1$. *It then follows from the earlier inequality that the firm will not invest in the periods after t.* □

Proposition 4. *Let K be the knowledge level at t. Then the firm is more likely to invest when* *a. N is larger,*

b. the interest rate r is smaller,

c. the systematic risk $\beta_C(K)$ *is smaller,*

d. the probability ϑ *that there will be no process innovation is larger.*

Proof of Proposition 4.

(a) Since $\Omega(t,K) \geq \Omega(t+1,K)$ *and* $M(t,K) > M(t+1,K)$, *we note that* $M(t,K)+\Omega(t,K)$ *is decreasing with t and thus increasing with N, from the definition of* $\Omega(t,K)$ and $M(t,K)$.

(b)From the definition of $M(t,K)$, *we can see that* $M(t,K)$ *is decreasing with r.*

We prove by induction that $M(t,K)+\Omega(t,K)$ *is also decreasing with r. If t=N,* $M(N,K)+\Omega(N,K) = M(N,K)$ *is decreasing with r.*

Suppose that $M(t+1,K)+\Omega(t+1,K)$ *is decreasing with r. Then* $M(t+1,K+1)+\Omega(t+1,K+1)$ *is also decreasing with r, since K is any finite non-negative integer.*

Then at time t, we have

$$M(t,K)+\Omega(t,K) =$$
$$M(t,K)+\max[M(t+1,K+1)+\Omega(t+1,K+1)-1,\ 0]-\max[M(t+1,K)+\Omega(t+1,K)-1,\ 0]+\Omega(t+1,K)$$

Since $M(t+1,K+1)+\Omega(t+1,K+1) > M(t+1,K)+\Omega(t+1,K)$, we have the following three cases:

Case 1: $M(t+1,K+1)+\Omega(t+1,K+1)-1<0$.

$$M(t,K)+\Omega(t,K) = M(t,K)+\Omega(t+1,K) = M(t,K)-M(t+1,K)+M(t+1,K)+\Omega(t+1,K)$$

is decreasing with r, since $[M(t+1,K)+\Omega(t+1,K)]$ *and* $[M(t,K)-M(t+1,K)]$ *are both decreasing with r.*

Case 2: $M(t+1,K+1)+\Omega(t+1,K+1)-1 \geq 0$ *and* $M(t+1,K)+\Omega(t+1,K)-1<0$.

$$M(t,K)+\Omega(t,K) = M(t,K)+M(t+1,K+1)+\Omega(t+1,K+1)-1+\Omega(t+1,K)$$

$$= M(t,K)-M(t+1,K)+M(t+1,K)+\Omega(t+1,K)+M(t+1,K+1)+\Omega(t+1,K+1)-1$$

is decreasing with r, since $[M(t+1,K)+\Omega(t+1,K)]$, $[M(t+1,K+1)+\Omega(t+1,K+1)]$

and $[M(t,K)-M(t+1,K)]$ *are all decreasing with r.*

Case 3: $M(t+1,K)+\Omega(t+1,K)-1\geq 0$

$M(t,K)+\Omega(t,K)=M(t,K)+M(t+1,K+1)+\Omega(t+1,K+1)-M(t+1,K)$ *is decreasing with r, since* $[M(t+1,K+1)+\Omega(t+1,K+1)]$ *and* $[M(t,K)-M(t+1,K)]$ *are both decreasing with r.*

We can use a similar approach as in (b) to prove (c) and (d). □

Lemma 5. *The present value of the incremental cash flows due to process improvement at time t,* $1\leq t\leq N$, *with the level of knowledge of K, is*

$$V_i(t,K)=I(t)\exp(\overline{C}_i(K)-\beta_C(K))\Pi[r,t],$$

where $i=1,2,3$, and $\Pi[r,t]=\sum_{s=t+1}^{N}e^{-(s-t)r}+\sum_{s=N+1}^{\infty}\vartheta^{s-N}e^{-(s-t)r}$ *is defined in equation* (A1) *and is the value of a perpetual consul bond, where the payment on this bond, after N, depreciates at a constant rate* $1-\vartheta$.

Proof of Lemma 5.

We use the same procedure used in the proof of Lemma 1 to prove each case and get the result. □

***Proposition* 5.** (*Optimal investment decision considering competitive factors*) *At any time t,* $1\leq t\leq N$**,** *with the level of knowledge K, if* $M(t,K)+\Omega(t,K)>1$, *then the firm should invest in process improvement; otherwise, the firm should not invest in process improvement, where* $M(t,K)$ *has been defined earlier and* $\Omega(t,K)$ *is calculated as follows:*

$$\Omega(t,K)=\sum_{j=1}^{N-t}W(t+j,K)=W(t+1,K)+\Omega(t+1,K) \qquad \text{(A7)}$$

$$W(t,K)=\max[M(t,K+1)+\Omega(t,K+1)-1,0]-\max[M(t,K)+\Omega(t,K)-1,0]+[L(t,K+1)-L(t,K)] \text{(A8)}$$

$$\Omega(N,K+j)=0,\quad j=0,\cdots,\ N-t+2. \qquad \text{(A9)}$$

Proof of Proposition 5.

When $t = N$, according to the definition in equation (A9), we have

$$\Omega(N,K) = \Omega(N,K+1) = \Omega(N,K+2) = 0$$

Now, when $t = N\text{-}1$, from the definition of the option to invest in the paper, we have

$$\Theta(N,K+1) = \max\left[H(N,K+1) + \Phi(N,K+1) - I(N), U(N,K+1)\right]$$

$$\Theta(N,K) = \max\left[H(N,K) + \Phi(N,K) - I(N), U(N,K)\right]$$

$\Theta(N,K+1) - \Theta(N,K) = I(N)W(N,K)$, where $W(N,K)$ is defined in (A8).

So

$$\Phi(N-1,K) = E_{N-1}\left[\frac{z(N)}{z(N-1)}\left[\Theta(N,K+1) - \Theta(N,K)\right]\right] = I(N-1)\Omega(N-1,K)$$

where $\Omega(N-1,K) = W(N,K)$ and the calculation of the expectation uses the lognormality of the pricing kernel and investment cost. Similarly, we have $\Omega(N-1,K+1) = W(N,K+1)$

When $t = N\text{-}2$,

$$\Theta(N-1,K+1) = \max\left[H(N-1,K+1) + \Phi(N-1,K+1) - I(N-1), U(N-1,K+1)\right]$$

$$\Theta(N-1,K) = \max\left[H(N-1,K) + \Phi(N-1,K) - I(N-1), U(N-1,K)\right]$$

$$\Theta(N-1,K+1) - \Theta(N-1,K) = I(N-1)W(N-1,K)$$

$$\begin{aligned}\Phi(N-2,K) &= E_{N-2}\left[\frac{z(N-1)}{z(N-2)}\left[\Theta(N-1,K+1) - \Theta(N-1,K)\right]\right] + E_{N-2}\left[\frac{z(N)}{z(N-2)}\left[\Theta(N,K+1) - \Theta(N,K)\right]\right] \\ &= I(N-2)\Omega(N-2,K)\end{aligned}$$

where $\Omega(N-2,K) = \sum_{j=1}^{2} W(N-2+j,K)$ and the law of iterated expectations (see Billingsley 1986, page 470) is used in calculating the second expectation.

When the current time period is t, we have

$$\Phi(t,K)=\sum_{j=1}^{N-t} E_t\left[\frac{z(t+j)}{z(t)}\left[\Theta(t+j,K+1)-\Theta(t+j,K)\right]\right]=I(t)\Omega(t,K)$$

where $\Omega(t,K)=\sum_{j=1}^{N-t} W(t+j,K)$. At time *t*, with the level of knowledge *K*, if

$H(t,K)+\Phi(t,K)-U(t,K)>I(N)$ or $M(t,K)+\Omega(t,K)>1$, the firm should go ahead and

invest. This proves the result. □

Proposition 6. *In an environment with uncertainty about the superiority of the firm's process relative to others in the industry**,** if K is the knowledge level at time t and the condition in (6) is satisfied, we have*

$M(t,K)+\Omega(t,K)>M(t+1,K)+\Omega(t+1,K), 1\le t<N..$

So, if the firm doesn't invest at time t, the firm will not invest in the remaining periods.

Proof of Proposition 6. We first show the following results when equation (6) is satisfied*:*

$$M(t,K)>M(t+1,K),\ 1\le t<N$$

$$\Omega(t,K)\ge\Omega(t+1,K),\ 1\le t<N$$

Since we can use similar approaches used in the proofs of the corresponding results without competitive factors in section 3 to show these results, the proof is omitted. Using these results, we have

$M(t,K)+\Omega(t,K)>M(t+1,K)+\Omega(t+1,K), 1\le t<N.$ □

Lemma 6. *Let K be the knowledge level at t. If* $V_1(t,K)+V_3(t,K)\ge V_2(t,K)$ *and the condition in equation (6) is satisfied, then the larger the value of* $\rho(t)$, *the larger is* $M(t,K)+\Omega(t,K)$ *and the more likely the firm is to invest in process improvement.*

Proof of Lemma 6. First, if $V_1(t,K)+V_3(t,K)\geq V_2(t,K)$, we can see that $M(t,K)$ is increasing with ρ. We prove by induction that $M(t,K)+\Omega(t,K)$ *is also increasing with* ρ. When $t = N$, we have $M(N,K)+\Omega(N,K)=M(N,K)$, which is increasing with ρ, since $\Omega(N,K)=0$ from the definition in equation (A9). Suppose that $M(t+1,K)+\Omega(t+1,K)$ is increasing with ρ at time period t +1. Now at *t*, according to equations (A7) and (A8), we have

$$
\begin{aligned}
&M(t,K)+\Omega(t,K)=M(t,K)+W(t+1,K)+\Omega(t+1,K)\\
&=M(t,K)+Max[M(t+1,K+1)+\Omega(t+1,K+1)-1,\ 0]-Max[M(t+1,K)+\Omega(t+1,K)-1,\ 0]+\\
&\left[L(t+1,K+1)-L(t+1,K)\right]+\Omega(t+1,K)
\end{aligned}
$$

Now we can use an approach similar to that in the proof of part (b) in Proposition 4 to show that that $M(t,K)+\Omega(t,K)$ *is increasing with* ρ, by considering that $\left[L(t+1,K+1)-L(t,K)\right]$, $\left[M(t+1,K)+\Omega(t+1,K)\right]$ and $\left[M(t,K)-M(t+1,K)\right]$ are all increasing with ρ. □